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The Particle Spectrum of Heterotic Compactifications

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Abstract

Techniques are presented for computing the cohomology of stable, holomorphic vector bundles over elliptically fibered Calabi-Yau threefolds. These cohomology groups explicitly determine the spectrum of the low energy, four-dimensional theory. Generic points in vector bundle moduli space manifest an identical spectrum. However, it is shown that on subsets of moduli space of co-dimension one or higher, the spectrum can abruptly jump to many different values. Both analytic and numerical data illustrating this phenomenon are presented. This result opens the possibility of tunneling or phase transitions between different particle spectra in the same heterotic compactification. In the course of this discussion, a classification of $SU(5)$ GUT theories within a specific context is presented.

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1 Introduction

The work of Hořava and Witten [1] opened the door to constructing phenomenologically realistic $N = 1$ supersymmetric vacua of strongly coupled heterotic string theory. A key ingredient in such constructions is the method presented in [2, 3, 4] and [5, 6] for finding stable, holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds. Using this technique, a large number of GUT theories with gauge groups $SU(5)$ and $SO(10)$ were produced in [5, 6, 7]. These ideas were generalized in [8, 9, 10], where it was shown how to construct stable, holomorphic vector bundles on torus-fibered Calabi-Yau threefolds. Using these generalized techniques, standard-like models were produced using Wilson lines to spontaneously break both $SU(5)$ [8] and $SO(10)$ [9] GUT groups. Within this context, a number of new phenomena were discussed such as small instanton phase transitions [11], non-perturbative superpotentials [12, 13], five-brane moduli space [14], brane worlds [15] and a new theory of the Big Bang [16].

One aspect of such theories that was left unsolved was how to compute the complete particle spectrum of the low-energy, four-dimensional theory. The existence of precisely three families of quarks and leptons was guaranteed in these theories by choosing the third Chern class of the holomorphic vector bundle appropriately. However, the number of other particles, such as the Higgs or exotic particles, was not specified. To compute their spectrum, it is necessary to construct the complete cohomology of the vector bundle V on the Calabi-Yau threefold X. This was carried out [17] in the case of the "standard embedding," that is, when $V = TX$, where TX is the tangent bundle. In this case, the spectrum is directly related to Betti numbers of TX, which are known. However, this is manifestly not the case for the bundles discussed above. For these more general vector bundles, the relevant cohomology groups are unrelated to the Dolbeault cohomology and, hence, are much more difficult to compute.

We discussed a general approach to this problem in [18]. It is the purpose of this paper to give explicit techniques for calculating the cohomology of stable, holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds. That is, we will show how to compute the complete spectrum for the low energy GUT theories that arise in this context. In the process of doing this, we have found what we believe to be an interesting phenomenon in particle physics. This is the following. Holomorphic vector bundles have complex moduli. In the present context, these have been discussed and enumerated in [12, 14, 19]. For generic values of these moduli, we find a specific particle spectrum. However, on loci of co-dimension one or higher in the vector bundle moduli space we find that the spectrum "jumps," changing abruptly by integer values. In this paper, we will conclusively demonstrate that this phenomenon exists. We will discuss the mathematical underpinnings of this result and give a concrete example using both analytic and numerical techniques.

Specifically, in this paper we will do the following. In section 2, we briefly review some salient facts about elliptically fibered Calabi-Yau threefolds. These spaces are fibered over base surfaces B, whose properties are discussed in Section 3. Section 4 is devoted to a short discussion of the method of constructing stable, holomorphic vector bundles from spectral data via the Fourier-Mukai transformation. In Section 5, we present the three physical constraints required of any phenomenologically relevant heterotic string vacuum. Using the mathematical constraints arising from these conditions, we present a classification of $SU(5)$ GUT theories that can arise in our context. This is given in Section 6. In Section 7, we present the general techniques for computing the low energy spectrum from the cohomology of the holomorphic vector bundle. First, we discuss the relationship between the spectrum and cohomology, as well as the constraints on the spectrum arising from the index theorem. We then present methods for computing the cohomology based on Leray spectral sequences and the Riemann-Roch theorem. For convenience, this is carried out within the context of vector bundles with an $SU(5)$ structure group. Section 8 is devoted to using these techniques to compute the complete cohomology of a specific $SU(5)$ GUT model satisfying the three physical constraints. The spectrum of this theory is then presented. We note and discuss the phenomenon that part of the cohomology and, hence, some of the spectrum is dependent upon the vector bundle moduli for which they are evaluated. Finally, in Section 9 we indicate why one expects the spectrum to be moduli dependent for general holomorphic bundles, as opposed to the standard embedding where this phenomenon does not occur. Appendices A and B present various aspects of topological data required in the text. Appendix C gives a general method for computing a large set of cohomology groups that are required in our discussion. Several matrices that are central to the calculation of the spectrum are defined and explicitly constructed in Appendix D, including their exact dependence on the vector bundle moduli. The spectra of these representations depend on the rank of one of these matrices. The rank is computed both analytically and numerically in Appendix E. Explicit data is presented, showing that the rank of this matrix is dependent upon where in moduli space it is evaluated.

Although much of this paper is presented within the context of $SU(5)$ GUT theories, the techniques introduced are completely general. They can be used to compute the spectrum of any heterotic vacuum.

2 Elliptically Fibered Calabi-Yau Threefolds

We will consider elliptically fibered threefolds X. Each such manifold has a base surface B and a mapping $\pi : X \to B$ such that $\pi^{-1}(b)$ is a smooth torus, E_b, for each generic point $b \in B$. Additionally, there are special points in the base over each of which the fiber is singular. These fibers are typically of type I_1, in the Kodaira classification, but may be more singular. What makes this torus fibration elliptic is the existence of a zero section; that is, there exists an analytic map $\sigma : B \to X$ that assigns to every element b of B an element $\sigma(b) \in E_b$. The point $\sigma(b)$ acts as the zero element for an Abelian group which turns E_b into an elliptic curve and X into an elliptic fibration. We will denote the fiber class by F.

In terms of explicit coordinates, one can express X as a Weierstrass model

$$y^2 z = x^3 + g_2 x z^2 + g_3 z^3 \tag{1}$$

which describes X as a divisor in a $\mathbb{P}^2$-bundle P over B. The coefficients g_2 and g_3 are sections of line bundles on the base. The bundle P is the projectivization $\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_B)$, where $\mathcal{L}$ is a line bundle on B which is the conormal bundle to the section σ. Subsequently, we have

$$x \sim \mathcal{O}_P(1) \otimes \mathcal{L}^2, \quad y \sim \mathcal{O}_P(1) \otimes \mathcal{L}^3, \quad z \sim \mathcal{O}_P(1) \tag{2}$$

and

$$g_2 \sim \mathcal{L}^4, \quad g_3 \sim \mathcal{L}^6 \ , \tag{3}$$

where we have used $\sim$ to denote “global section of.’

An important property of elliptic fibrations is that X has a $\mathbb{Z}_2$ symmetry $\tau = (-1)_X$, which, on the Weierstrass coordinates defined in (1) acts as

$$\tau : y \to -y \tag{4}$$

while leaving x and z invariant. Clearly this action leaves the Weierstrass equation (1) unchanged. In other words τ is a natural involution on X. It acts trivially on the base B and maps each element $b \in E_b$ to its inverse $-b$.

$N = 1$ supersymmetry in four-dimensions demands that X be a Kahler manifold with vanishing first Chern class of its tangent bundle TX; that is,

$$c_1(TX) = 0 \ . \tag{5}$$

Such manifolds always admit a Kahler metric of $SU(3)$ holonomy and are called Calabi-Yau manifolds. Henceforth, we will choose X to be a Calabi-Yau threefold. In general, the Chern classes of X can be conveniently expressed in terms of those of the base B [3, 20]. In addition to (5), one finds that

$$c_2(TX) = 12\sigma \cdot \pi^*(c_1(TB)) + \pi^*(c_2(TB) + 11c_1^2(TB)),$$
$$c_3(TX) = -60(c_1(TB)^2 \cdot B)\text{pt} \ , \tag{6}$$

where $c_1(TB)$ and $c_2(TB)$ are the first and second Chern classes of B respectively and pt is the class of a point. When X is a Calabi-Yau threefold, severe restrictions are placed on the base surface B. It turns out that B can only be Enriques, del Pezzo, Hirzebruch and blowups of Hirzebruch surfaces [21]. We will present some relevant properties of these surfaces shortly.

Throughout this paper we will make frequent use of the intersection relation

$$\sigma \cdot \sigma = -\pi^*(c_1(TB)) \cdot \sigma \ , \tag{7}$$

which follows from the adjunction formula.

3 Properties of the Base Surface

We now present the requisite properties, such as Chern classes and homology groups, of the surfaces B. Before doing so, however, it is helpful to define some fundamental notions.

Consider a complex surface B and its second homology group $H_2(B, \mathbb{Z})$. Let $C \subset B$ be a holomorphic curve in B and $[C] \in H_2(B, \mathbb{Z})$ the class of curves equivalent to C. Then $[C]$ is called an "effective" class. Clearly, not every class, such as $-[C]$, is effective. If $[C]$ and $[D]$ are two effective classes, then so is $m[C] + n[D]$ where $m, n \in \mathbb{Z}_{\geq 0}$. Therefore, the subset of effective classes forms a cone in $H_2(B, \mathbb{Z})$, called the Mori cone. The Mori cone is spanned by a countable number of generators, $[C_i]$, where $C_i \subset B$ are irreducible curves. That is, any effective class $[C]$ can be expressed as

$$[C] = \sum_i r_i [C_i], \quad r_i \in \mathbb{Z}_{\geq 0} \ . \tag{8}$$

The reader is referred to [22, 23, 24], for example, for details. The Mori cone is not necessarily finitely generated over $\mathbb{Z}_{\geq 0}$, although for all surfaces discussed below, with the exception of $d\mathbb{P}_9$, the associated Mori cones are indeed finitely generated. We will shortly present their bases explicitly. For $d\mathbb{P}_9$, the Mori cone has an infinite number of generators. Nevertheless, there is a convenient description of them.

Let $C \subset B$ be a holomorphic curve in a complex surface B. Since C is a divisor of B, there exists a line bundle $\mathcal{O}_B(C)$ which has a section s_C, unique up to scalar multiplication, whose zero locus is C. Now, consider another curve $C' \neq C$ with the property that $\mathcal{O}_B(C') \simeq \mathcal{O}_B(C)$. Then, there exists a section $s_{C'}$ of $\mathcal{O}_B(C)$ whose zero locus is C'. Note that $s_C/s_{C'}$ is a meromorphic function f on B. Two such divisors C and C' are said to be linearly equivalent. The set of all divisors linearly equivalent to C, denoted by $|C|$, is called the linear system associated with C. A crucial property of linear systems is the following. A base point of a linear system $|C|$ of curves on B is the intersection of all its members. If there is no such common point, then $|C|$ is called base point free. Furthermore, note that all numerical properties of a divisor C, such as its self-intersection number, are completely determined by its linear system.

We have restricted the discussion in this section to divisor classes $[C]$, divisors C, and to bundles $\mathcal{O}_B(C)$ associated with B. However, all of our remarks apply to classes, divisors and line bundles of any complex manifold, such as the threefold X.

3.1 Hirzebruch Surfaces

The Hirzebruch surfaces $\mathbb{F}_r$ are $\mathbb{P}^1$ fibrations over $\mathbb{P}^1$. There is an infinite family of such surfaces indexed by $r \in \mathbb{Z}_{\geq 0}$. The second homology group is

$$H_2(\mathbb{F}_r, \mathbb{Z}) = \text{span}_{\mathbb{Z}} \{S, \ \mathcal{E}\} \ , \tag{9}$$

where the generators S and $\mathcal{E}$ are effective classes with the intersection numbers

$$S \cdot S = -r, \quad \mathcal{E} \cdot \mathcal{E} = 0, \quad S \cdot \mathcal{E} = 1 \ . \tag{10}$$

All effective classes are of the form

$$aS + b\mathcal{E}, \qquad a, b \in \mathbb{Z}_{\geq 0} \ . \tag{11}$$

The aggregate of these is called the Mori cone of $\mathbb{F}_r$. The Chern classes are given by

$$\begin{aligned} c_1(T\mathbb{F}_r) &= -c_1(K_{\mathbb{F}_r}) = 2S + (r+2)\mathcal{E} \\ c_2(T\mathbb{F}_r) &= 4 \ , \end{aligned} \tag{12}$$

where $K_{\mathbb{F}_r}$ is the canonical bundle. Finally, on $\mathbb{F}_r$, the linear system $|aS + b\mathcal{E}|$ is base-point free if

$$b \geq a \ r \ . \tag{13}$$

3.2 del Pezzo Surfaces

There are, in all, nine del Pezzo surfaces, which we denote as $d\mathbb{P}_r$ for $r = 1, \ldots, 9$. Each $d\mathbb{P}_r$ is the $\mathbb{P}^2$ surface blown up at r generic points. The second homology group for $d\mathbb{P}_r$ is

$$H_2(d\mathbb{P}_r, \mathbb{Z}) = \text{span}_{\mathbb{Z}} \{\ell,\ E_{i=1,\ldots,r}\} \ , \tag{14}$$

where ℓ is the hyperplane class in $\mathbb{P}^2$ and $E_{i=1,\ldots,r}$ are the r exceptional divisors. Each E_i corresponds to the $\mathbb{P}^1$ blowup of a point in $\mathbb{P}^2$. These classes have the following intersections

$$\ell \cdot \ell = 1, \quad \ell \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij} \ . \tag{15}$$

The Chern classes are given by

$$\begin{aligned} c_1(Td\mathbb{P}_r) &= -c_1(K_{d\mathbb{P}_r}) = 3\ell - \sum_{i=1}^{r} E_i \\ c_2(Td\mathbb{P}_r) &= r + 3 \ , \end{aligned} \tag{16}$$

where $K_{d\mathbb{P}_r}$ is the canonical bundle. We now study the Mori cone of $d\mathbb{P}_r$. The effective classes in $H_2(d\mathbb{P}_r, \mathbb{Z})$, that is, those which can be expressed as non-negative integral combinations of classes of irreducible curves, are tabulated in [25]. Here, we re-cast them into a more convenient form and present the generators for the Mori cones in Table 1.

We note that $d\mathbb{P}_9$ is itself an elliptic fibration over $\mathbb{P}^1$ with fiber class

$$f = c_1(Td\mathbb{P}_9) \ . \tag{17}$$

As stated in Table 1, this class is a generator of the Mori cone of $d\mathbb{P}_9$. It is useful to note from (15), (16) and (17) that

$$f^2 = 0 \ . \tag{18}$$

The remaining generators, y_i, of $d\mathbb{P}_9$ form an infinite, but countable, set whose properties are listed in Table 1.

We will also need the base-point free condition for linear systems on del Pezzo surfaces. Here, we are aided by Proposition 2.3 of [27] which states the following. Let η be a divisor on a del Pezzo surface $d\mathbb{P}_r$ for $2 \leq r \leq 7$ such that $\eta \cdot E \geq 0$ for every curve E for which $E \cdot E = -1$ and $E \cdot c_1(Td\mathbb{P}_r) = 1$. Then the linear system $|\eta|$ is base point free. Note from Table 1 that the bases for the Mori cones are precisely all curves satisfying the two conditions $E \cdot E = -1$ and $E \cdot c_1(Td\mathbb{P}_r) = 1$. As a last remark, note that the surface $d\mathbb{P}_1$ is actually isomorphic to $\mathbb{F}_1$. One can see this, for example, from the fact that $d\mathbb{P}_1$ fibers over its unique exceptional divisor S where $S^2 = -1$ and each fiber is a $\mathbb{P}^1$. Alternatively, these two surfaces are toric varieties with identical toric diagrams.

r	Generators	Distinct Indices	Number
1	$E_1,\ \ell - E_1$		2
2	$E_i,\ \ell - E_1 - E_2$	$i = 1,\ 2$	3
3	$E_i, \ell - E_i - E_j$	$i, j = 1, 2, 3$	6
4	$E_i,\ \ell - E_i - E_j$	$i, j = 1, \ldots, 4$	10
5	$E_i,\ \ell - E_i - E_j,\ 2\ell - E_i - E_j - E_k - E_l - E_m$	$i, j, k, l, m = 1, \ldots, 5$	16
6	$E_i,\ \ell - E_i - E_j,\ 2\ell - E_i - E_j - E_k - E_l - E_m$	$i, j, k, l, m = 1, \ldots, 6$	27
7	$E_i,\ \ell - E_i - E_j,\ 2\ell - E_i - E_j - E_k - E_l - E_m,$ $3\ell - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o$	$i, j, k, l, m, n, o = 1, \ldots, 7$	56
8	$E_i,\ \ell - E_i - E_j,\ 2\ell - E_i - E_j - E_k - E_l - E_m,$ $3\ell - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o,$ $4\ell - 2(E_i + E_j + E_k) - \sum_{i=1}^{5} E_{m_i},$ $5\ell - 2\sum_{i=1}^{6} E_{m_i} - E_k - E_l,\ 6\ell - 3E_i - 2\sum_{i=1}^{7} E_{m_i}$	i, j, k, l, m, n, o, m_i $= 1, \ldots, 8$	240
9	$f = 3\ell - \sum_{i=1}^{9} E_i,$ and $\{y_i\}$ such that $y_i^2 = -1,\ y_i \cdot f = 1$	—	∞

Table 1: The generators for the Mori cone of a generic $d\mathbb{P}_r$ for $r = 1, \ldots, 9$. All effective classes of curves in $H_2(d\mathbb{P}_r, \mathbb{Z})$ can be written as non-negative integral combinations of these generators. We emphasize that all indices are distinct.

3.3 Enriques Surfaces

The Enriques surface $\mathbb{E}$ is obtained from a K3 surface modulo involutions. Its canonical bundle is torsion, that is,

$$K_{\mathbb{E}} \otimes K_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}} \ . \tag{19}$$

This implies that

$$2c_1(T\mathbb{E}) = 0 \ . \tag{20}$$

The second Chern class is given by

$$c_2(T\mathbb{E}) = 12 \ . \tag{21}$$

The second homology group is

$$H_2(\mathbb{E}, \mathbb{Z}) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}_2 \ . \tag{22}$$

We will not present the explicit generators for the Mori cone of $\mathbb{E}$ since these will not be used in this paper.

4 Vector Bundles on Elliptically Fibered Calabi-Yau Threefolds

We consider rank n stable holomorphic vector bundles V on X. These bundles have a convenient description, known as the spectral cover construction [2, 3, 4, 5, 6]. The spectral data is given by two objects, an effective divisor $\mathcal{C}_V$ of X, called the spectral cover, and a spectral line bundle $\mathcal{N}_V$ on $\mathcal{C}_V$. The spectral cover, $\mathcal{C}_V$, is a surface in X that is an n-fold cover, $p : \mathcal{C}_V \rightarrow B$, of the base B. Its general form is

$$\mathcal{C}_V \in |n\sigma + \pi^*\eta| \ , \tag{23}$$

where σ is the zero section associated with π, and η is some effective curve in B. The spectral line bundle $\mathcal{N}_V$ is defined by its first Chern class

$$c_1(\mathcal{N}_V) = n(\frac{1}{2} + \lambda)\sigma + (\frac{1}{2} - \lambda)\pi^*\eta + (\frac{1}{2} + n\lambda)\pi^* c_1(TB) \ , \tag{24}$$

where λ is a rational number such that

$$\begin{array}{rclll} \lambda & = & m, & & n \ \text{even} \\ \lambda & = & m + \frac{1}{2}, & & n \ \text{odd} \end{array} \tag{25}$$

for some $m \in \mathbb{Z}$. When n is even, we must also impose that

$$\eta = c_1(TB) \bmod 2 \ . \tag{26}$$

Note that, as defined in (24), $\mathcal{N}_V$ is actually a line bundle on X. It can, of course, be restricted to be a line bundle over $\mathcal{C}_V$. Throughout this paper, we will, in general, not distinguish between $\mathcal{N}_V$ and $\mathcal{N}_V|_{\mathcal{C}_V}$, denoting both by $\mathcal{N}_V$.

4.1 Fourier-Mukai Transformation

Given the spectral data $(\mathcal{C}_V, \mathcal{N}_V)$, one can construct the vector bundle V explicitly, using the Fourier-Mukai transformation

$$(\mathcal{C}_V, \mathcal{N}_V) \stackrel{FM}{\longleftrightarrow} V \ . \tag{27}$$

We briefly remind the reader of the structure of this transformation [2, 3, 4, 5, 6]. Let us form the fiber product $X \times_B X'$, where $X' \simeq X$ is another copy of X. We let $\pi : X \rightarrow B$

and $\pi' : X' \to B$ be the projections onto the base B with sections σ and σ' respectively. The fiber product is a four-dimensional space defined as

$$X \times_B X' = \{(x, x') \in X \times X' \mid \pi(x) = \pi'(x')\} \ . \tag{28}$$

Therefore, over any generic point $b \in B$ we have a fiber $E_b \times E'_b$, where E_b and E'_b are elliptic curves. We define the Poincaré sheaf $\mathcal{P}$ to be

$$\mathcal{P} = \mathcal{O}_{X \times_B X'}(\Delta - \sigma \times_B X' - X \times_B \sigma') \otimes K_B \ , \tag{29}$$

where Δ is the (diagonal) divisor given by the set of points (x, x) in $X \times_B X'$. Recall [8] that $\mathcal{P}$ is a bundle except at points $(x, x') \in X \times_B X'$ where both x and x' are singular points of their respective fibers.

Now, let us take the spectral cover $\mathcal{C}_V \subset X$ and form the fiber product $\mathcal{C}_V \times_B X'$. Then, we have the following diagram, with projection maps π_1 and π_2 appropriately defined.

$$\begin{array}{ccccc} \mathcal{N}_V & & & & \\ \downarrow & & & & \\ \mathcal{C}_V & \stackrel{\pi_2}{\longleftarrow} & \mathcal{C}_V \times_B X' & \stackrel{\pi_1}{\longrightarrow} & X' \end{array} \tag{30}$$

The Fourier-Mukai transformation is the explicit map that re-constructs the vector bundle V from the spectral data in accordance with (30)

$$V = \pi_{1*}\left((\pi_2^* \mathcal{N}_V) \otimes \mathcal{P}\right) \ . \tag{31}$$

We emphasize that we are using, as it is standard in the literature, the canonical isomorphism of $X \simeq X'$ so that saying V is a vector bundle on X' is equivalent to saying that V is a vector bundle on X. In the same way, we could have defined the spectral data $(\mathcal{C}'_V, \mathcal{N}'_V)$ in X' and produced a vector bundle V on X.

Altogether, we have the following commutative diagram

$$\begin{array}{ccc} \pi_2^* \mathcal{N}_V \otimes \mathcal{P} & \stackrel{\pi_{1*}}{\longrightarrow} & V \\ \downarrow & & \downarrow \\ \mathcal{C}_V \times_B X' & \stackrel{\pi_1}{\longrightarrow} & X' \\ \pi_2 \downarrow & & \downarrow \pi' \\ \mathcal{C}_V & \stackrel{\pi_{\mathcal{C}}}{\longrightarrow} & B \end{array} \quad . \tag{32}$$

The Chern classes of V are found to be [2, 3, 4]

$$\begin{aligned} c_1(V) &= 0, \\ c_2(V) &= \sigma \cdot \pi^*(\eta) - \frac{1}{24} c_1(TB)^2 (n^3 - n) F + \frac{1}{2}(\lambda^2 - \frac{1}{4}) n \ \eta \cdot (\eta - n \ c_1(TB)) F, \\ c_3(V) &= 2\lambda\eta \cdot (\eta - n \ c_1(TB)) \mathrm{pt} \ . \end{aligned} \tag{33}$$

5 The Physical Constraints

The requirements of particle physics phenomenology put three strong constraints on both the Calabi-Yau threefold X and the holomorphic vector bundle V. These arise from the necessity that the theory be consistent quantum mechanically, that there be three families of quarks and leptons and that the theory preserve $N = 1$ supersymmetry. Let us examine each of these constraints.

5.1 Anomaly Cancellation Condition

The anomaly cancellation condition is given by

$$c_2(TX) - c_2(V) = W \ , \tag{34}$$

where W is the five-brane class in the vacuum. Furthermore, since these must be physical five-branes, the class W must be effective, that is, in the Mori cone of $H_2(X, \mathbb{Z})$. The five-brane class can be written as

$$W = W_B + a_F F \ , \tag{35}$$

where, using (6), (33) and (34),

$$\begin{aligned} W_B &= \sigma \cdot \pi^*(12 c_1(TB) - \eta) \ , \\ a_F &= c_2(TB) + (11 + \frac{n^3 - n}{24}) c_1(TB)^2 - \frac{1}{2} n (\lambda^2 - \frac{1}{4}) \eta \cdot (\eta - n\ c_1(TB)) \ . \end{aligned} \tag{36}$$

It was shown [5, 6] that W is an effective class in X if and only if

$$W_B \text{ is effective in } B, \quad a_F \geq 0 \ . \tag{37}$$

Henceforth, we will require the expressions in (36) to satisfy the constraints given in (37).

5.2 Three Family Models

The number of generations, N_{gen}, is related to the zero-modes of the Dirac operator $\not{D}$ in the presence of the vector bundle. Specifically, it is given by

$$N_{gen} = \text{index}(V, \not{D}) = \int_X \text{td}(TX)\text{ch}(V) = \frac{1}{2} \int_X c_3(V) \ , \tag{38}$$

where we have used the index theorem. We are interested in three-family models, that is theories for which

$$N_{gen} = 3 \ . \tag{39}$$

It then follows from (33), (38) and (39) that

$$3 = \lambda\eta \cdot (\eta - n\ c_1(TB))\ . \tag{40}$$

We will, henceforth, require that this constraint be satisfied. Note that (40) simplifies the expression for a_F in (36). It follows that the condition on a_F in (37) becomes

$$a_F = c_2(TB) + (11 + \frac{n^3 - n}{24})c_1(TB)^2 - \frac{3}{2\lambda}n(\lambda^2 - \frac{1}{4}) \geq 0\ . \tag{41}$$

5.3 Irreducibility of the Spectral Cover

In order for the gauge connection to preserve $N = 1$ supersymmetry, it must satisfy the hermitian Yang-Mills equation. The theorems of Uhlenbeck and Yau [28] and Donaldson [29] state that a rank n holomorphic vector bundle V will admit such a connection if V is stable. We will, therefore, impose stability as a constraint on V. It was shown in [2, 3, 4] that if V is constructed via a Fourier-Mukai transformation from spectral data, then V will be stable if the spectral cover is irreducible[1]. What are the conditions on the linear system $|\mathcal{C}_V|$ such that it contains an irreducible divisor? This will be the case when the following two conditions are met [11].

$$\begin{array}{ll} (1) & |\pi^*\eta| \text{ contains an irreducible divisor in } X\ , \\ (2) & c_1(\pi^* K_B^{\otimes n} \otimes \mathcal{O}_X(\pi^*\eta)) \text{ is effective in } H_4(X, \mathbb{Z})\ . \end{array} \tag{42}$$

We will satisfy these two conditions if we impose

$$\begin{array}{ll} (1) & \text{The linear system } |\eta| \text{ is base-point free in } B\ , \\ (2) & \eta + nc_1(K_B) \text{ is effective in } B\ . \end{array} \tag{43}$$

In order to preserve $N = 1$ supersymmetry, the conditions in (43) must be imposed in addition to (37) and (40).

5.4 Summary of the Constraints

For clarity, we summarize here the physical constraints, namely (37), (40), and (43), which we impose on our vector bundle. These become the following conditions on the effective base

[1] More precisely, the notion of stability requires the choice of an ample class $H \in H^2(X, \mathbb{Z})$ and, in the situation that the cover is irreducible, we can find an ample class H such that V is stable.

curve η as well as on the parameter λ.

$$
\begin{array}{llll}
(1) & W_B \text{ effective} & : & 12c_1(TB) - \eta \quad \text{is effective,} \\
(2) & a_F > 0 & : & c_2(TB) + (11 + \frac{n^3-n}{24})c_1(TB)^2 - \frac{3}{2\lambda}n(\lambda^2 - \frac{1}{4}) \geq 0, \\
(3) & \text{Three families} & : & \lambda\, \eta \cdot (\eta - n\ c_1(TB)) = 3, \qquad \lambda \in \begin{cases} \mathbb{Z}, & n \text{ even} \\ \mathbb{Z}/2, & n \text{ odd,} \end{cases} \\
(4) & \text{Stability of } V & : & |\eta| \text{ is base-point free,} \\
(5) & \text{Stability of } V & : & \eta - nc_1(TB) \quad \text{is effective .}
\end{array}
\tag{44}
$$

In this paper, for specificity, we will restrict our attention to the case

$$
n = 5 \ . \tag{45}
$$

This corresponds to constructing an $SU(5)$ GUT model at low energy. Because $n = 5$ is odd, λ has to be half integral by (25). Thus, the third condition in (44) implies, since $\eta \cdot (\eta - n\ c_1(TB))$ is integral, that the only possibilities for λ are

$$
\lambda = \pm\frac{1}{2},\ \pm\frac{3}{2}\ . \tag{46}
$$

Therefore, (44) simplifies to the constraints

$$
\begin{array}{ll}
(1) & 12c_1(TB) - \eta \quad \text{is effective,} \\
(2) & c_2(TB) + 16c_1(TB)^2 - \frac{15}{2\lambda}(\lambda^2 - \frac{1}{4}) \geq 0, \\
(3) & \lambda\, \eta \cdot (\eta - 5\ c_1(TB)) = 3, \qquad \lambda = \pm\frac{1}{2} \text{ or } \pm\frac{3}{2}\ , \\
(4) & |\eta| \text{ is base-point free,} \\
(5) & \eta - 5c_1(TB) \quad \text{is effective .}
\end{array}
\tag{47}
$$

6 A Classification of $SU(5)$ GUT Theories from Heterotic M-Theory Compactification

6.1 Eliminating the Enriques Surfaces

In [6] it was shown that Enriques surfaces will never satisfy the first condition in constraint (37), that is, condition (1) in (44) and (47). We briefly present a simplified version of this argument. Recalling the expression for W_B in (36), we have

$$
W_{\mathbb{E}} = \sigma \cdot \pi^*(12c_1(T\mathbb{E}) - \eta)\ . \tag{48}
$$

Furthermore, from (19), we know that

$$K_{\mathbb{E}}^{\otimes 12} = \mathcal{O}_{\mathbb{E}} \tag{49}$$

because 12 is even. Therefore, expression (48) then becomes

$$W_{\mathbb{E}} = -\sigma \cdot \pi^* \eta \ . \tag{50}$$

Since η is an effective class, it follows that $W_{\mathbb{E}}$ can be effective only if η is trivial. This, of course, would violate the three-family condition (3) of (47). We conclude that the Enriques surface is not consistent with the anomaly cancellation and three-family conditions.

6.2 $d\mathbb{P}_9$ Surfaces

We now show that the generic $d\mathbb{P}_9$ is ruled out as well. Recall from (17) that

$$f = c_1(Td\mathbb{P}_9) \tag{51}$$

is the fiber class over $\mathbb{P}^1$. It then follows from condition (1) of (47) that

$$W_{d\mathbb{P}_9} = 12f - \eta \tag{52}$$

must be effective. Since we are considering generic $d\mathbb{P}_9$, we can use the results in Table 1 and show that

$$12f - \eta = \alpha f + \sum_i \beta_i y_i \ , \quad \text{for some } \alpha, \beta_i \in \mathbb{Z}_{\geq 0} \ , \tag{53}$$

where y_i are such that

$$y_i^2 = -1, \ \ y_i \cdot f = 1 \ . \tag{54}$$

We remark that, for a non-generic $d\mathbb{P}_9$, expressions (53) and (54) need not be valid. Now, by (18) and (54),

$$(12f - \eta) \cdot f = -\eta \cdot f = (\sum_i \beta_i y_i) \cdot f = \sum_i \beta_i \geq 0 \ . \tag{55}$$

On the other hand, η must be effective, so we can write

$$\eta = \alpha' f + \sum_j \beta_j' y_j \ , \quad \text{for some } \alpha', \beta_j' \in \mathbb{Z}_{\geq 0} \ . \tag{56}$$

It follows that

$$\eta \cdot f = \sum_j \beta_j' \geq 0 \ . \tag{57}$$

Combining (55) and (57), we have

$$\eta \cdot f = -\sum_i \beta_i = \sum_j \beta'_j \ . \tag{58}$$

However, all β_i and β'_j are non-negative. Therefore, we must have

$$\beta_i = \beta'_j = 0 \ , \tag{59}$$

which implies that

$$\eta = \alpha' f \ . \tag{60}$$

That is, η is proportional to the fiber class. Finally, the three-family condition (3) of (47) requires that

$$\lambda \eta \cdot (\eta - n\ f) = 3 \ . \tag{61}$$

However, the left hand side of (61) vanishes due to (17) and (60). It follows that on a generic $d\mathbb{P}_9$ surface the effectiveness condition for five-branes and the three family constraint are in contradiction. Therefore, to obtain phenomenologically acceptable theories of this type, one must consider special non-generic $d\mathbb{P}_9$ base surfaces. See, for example, [8, 9].

6.3 Hirzebruch Surfaces

The remaining possibilities for the base surfaces are then the Hirzebruch surfaces $\mathbb{F}_r$, certain blowups of these surfaces and the del Pezzo surfaces $d\mathbb{P}_r$ for $r = 1, \ldots, 8$. In this paper, we will not discuss the blowups of $\mathbb{F}_r$. Let us first consider the case of $\mathbb{F}_r$. Using (11), we can write the effective class of the base curve in the spectral cover (23) as

$$\eta = aS + b\mathcal{E}, \qquad a, b \in \mathbb{Z}_{\geq 0} \ . \tag{62}$$

Next, we must satisfy the five requirements in (47). Recalling the Chern classes from (12) and the intersection numbers from (10), these translate into the following conditions for $a, b, r \in \mathbb{Z}_{\geq 0}$ and $\lambda = \pm\frac{1}{2},\ \pm\frac{3}{2}$.

$$\begin{aligned}
&(1) \qquad 24 - a \geq 0\ , \quad 12(r+2) - b \geq 0, \\
&(2) \qquad 44 - \frac{5}{2\lambda}(\lambda^2 - \frac{1}{4}) \geq 0, \\
&(3) \qquad \lambda(2\,a\,b - 10\,a - 10\,b - a^2\,r + 5a\,r) = 3, \\
&(4) \qquad b \geq a\ r, \\
&(5) \qquad a - 10 \geq 0\ , \quad b - 5(r+2) \geq 0\ .
\end{aligned} \tag{63}$$

Note that in the fourth condition in (63) we have used (13). The five expressions in (63) constitute a system of Diophantine inequalities. We have studied these inequalities for the four allowed values of λ and found that only

$$\lambda = \frac{1}{2} \tag{64}$$

permits solutions. Subject to this constraint, the only solutions are

$$(a, b, r) = (12, 15, 1),\ (13, 15, 1)\ . \tag{65}$$

That is,

$$B = \mathbb{F}_1, \quad \eta = 12S + 15\mathcal{E},\ 13S + 15\mathcal{E}\ . \tag{66}$$

We see that our physical conditions are so stringent that they restrict the Hirzebruch surfaces to $\mathbb{F}_1$ and the possible spectral covers on it to those specified by the two curves in (66).

6.4 The $d\mathbb{P}_2$ Surface

Let us now consider the del Pezzo surfaces. Since, as we remarked at the end of Subsection 3.2, $d\mathbb{P}_1 \simeq \mathbb{F}_1$, we can start with the next surface in the del Pezzo series, namely, $d\mathbb{P}_2$. Referring to Table 1, we write the effective base curve η as

$$\eta = aE_1 + b(\ell - E_1 - E_2) + cE_2\ , \qquad a, b, c \in \mathbb{Z}_{\geq 0}\ . \tag{67}$$

Next, we must satisfy the constraints (47). The difficult constraint to satisfy is condition (4), which requires that the linear system $|\eta|$ be base-point free. Using the theorem stated in Subsection 3.2 we have that the base-point-free condition (4) of (47) becomes, for $d\mathbb{P}_2$,

$$\eta \cdot E_1 \geq 0\ , \quad \eta \cdot (\ell - E_1 - E_2) \geq 0\ , \quad \eta \cdot E_2 \geq 0\ . \tag{68}$$

Substituting expression (67) for η and using the intersection relations (15), the condition becomes

$$b - a \geq 0\ , \quad a - b + c \geq 0\ , \quad b - c \geq 0\ . \tag{69}$$

Using (16) and (69), the full set of constraints in (47) explicitly becomes a system of Diophantine inequalities for the parameters $a, b, c \in \mathbb{Z}_{\geq 0}$ in (67) and $\lambda = \pm\frac{1}{2},\ \pm\frac{3}{2}$. They are

$$\begin{array}{ll}
(1) & 24 \geq a,\ 36 \geq b,\ 24 \geq c, \\
(2) & 39 - \frac{5}{2\lambda}(\lambda^2 - \frac{1}{4}) \geq 0, \\
(3) & \lambda(-a^2 + 2ab - b^2 + 2bc - c^2 - 5a - 5b - 5c) = 3, \\
(4) & b - a \geq 0,\ a - b + c \geq 0,\ b - c \geq 0, \\
(5) & a \geq 10,\ b \geq 15,\ c \geq 10.
\end{array} \tag{70}$$

We have studied these equations numerically for the four allowed values of λ. Once again, we find that only $\lambda = \frac{1}{2}$ is allowed. For this λ, the allowed values of a, b, c are found to be

$$(a,b,c) \;=\; (10,15,12),(10,15,13),(10,17,10),(10,18,10),(12,15,10),(13,15,10). \quad (71)$$

The six solutions in (71) correspond to the following classes

$$\begin{aligned}\eta \;=\; & 15\ell - 5E_1 - 3E_2, \quad 15\ell - 5E_1 - 2E_2, \quad 17\ell - 7E_1 - 7E_2, \\ & 18\ell - 8E_1 - 8E_2, \quad 15\ell - 3E_1 - 5E_2, \quad 15\ell - 2E_1 - 5E_2. \end{aligned} \quad (72)$$

We conclude that for $B = d\mathbb{P}_2$, exactly six types of vector bundles corresponding to the curves in (72) will satisfy all of the physical constraints.

6.5 $d\mathbb{P}_3$ Surface

We move on to the third del Pezzo surface. Again, referring to Table 1, we write the effective base class η as

$$\eta = n_1E_1 + n_2(\ell - E_1 - E_2) + n_3E_2 + n_4(\ell - E_1 - E_3) + n_5(\ell - E_2 - E_3) + n_6E_3 \;, \qquad n_i \in \mathbb{Z}_{\geq 0} \;. \quad (73)$$

As above, we can use the theorem in Subsection 3.2 to obtain the following conditions for the linear system $|\eta|$ to be base-point free. They are

$$\begin{aligned} -n_1 + n_2 + n_4 \geq 0, \quad n_1 - n_2 + n_3 \geq 0, \quad n_2 - n_3 + n_5 \geq 0, \\ n_1 - n_4 + n_6 \geq 0, \quad n_3 - n_5 + n_6 \geq 0, \quad n_4 + n_5 - n_6 \geq 0 \;. \end{aligned} \quad (74)$$

The conditions (1) and (5) for effectiveness in (47) are now more complicated than previously. For example, condition (1) becomes

$$\begin{aligned} 12c_1(Td\mathbb{P}_3) - \eta \;=\; & (36 - n_2 - n_4 - n_5)\ell + (-12 - n_1 + n_2 + n_4)E_1 + \\ & (-12 + n_2 - n_3 + n_5)E_2 + (-12 + n_4 + n_5 - n_6)E_3 \;. \end{aligned} \quad (75)$$

This can be written as

$$12c_1(Td\mathbb{P}_3) - \eta = \begin{pmatrix} 36 - a_3 - a_6 - n_1 - n_3 - n_6 \\ 24 - a_6 - n_2 - n_6 \\ a_3 \\ 24 - a_3 - n_3 - n_4 \\ -12 + a_3 + a_6 + n_3 - n_5 + n_6 \\ a_6 \end{pmatrix} \cdot v \;, \quad (76)$$

where

$$v = \{E_1, \ell - E_1 - E_2, E_2, \ell - E_1 - E_3, \ell - E_2 - E_3, E_3\} \tag{77}$$

is the basis of the Mori cone of $d\mathbb{P}_3$ and $a_3, a_6 \in \mathbb{Z}$ are arbitrary parameters. Similarly, condition (2) becomes

$$\eta - 5c_1(Td\mathbb{P}_3) = \begin{pmatrix} -10 - b_5 + n_1 + n_5 \\ -10 - b_6 + n_2 + n_6 \\ -5 + b_5 - b_6 + n_3 - n_5 + n_6 \\ -5 - b_5 + b_6 + n_4 + n_5 - n_6 \\ b_5 \\ b_6 \end{pmatrix} \cdot v \tag{78}$$

for arbitrary parameters $b_3, b_6 \in \mathbb{Z}$. Effectiveness of the two classes $\eta - 5c_1(Td\mathbb{P}_3)$ and $12c_1(Td\mathbb{P}_3) - \eta$ means that the components of the two vectors in brackets must be non-negative for at least one choice of the parameters a_3, a_6 and b_3, b_6 respectively. Using these results and (74), the five conditions in (47) for $d\mathbb{P}_3$ translates into the following system of Diophantine inequalities for $n_{i=1,\ldots 6} \in \mathbb{Z}_{\geq 0}$, $a_3, a_6, b_5, b_6 \in \mathbb{Z}$ and $\lambda = \pm\frac{1}{2}$ or $\pm\frac{3}{2}$. They are

$$\begin{array}{ll}
(1) & \begin{array}{l} 36 - a_3 - a_6 - n_1 - n_3 - n_6 \geq 0,\ 24 - a_6 - n_2 - n_6 \geq 0,\ a_3 \geq 0, \\ 24 - a_3 - n_3 - n_4 \geq 0,\ -12 + a_3 + a_6 + n_3 - n_5 + n_6 \geq 0,\ a_6 \geq 0, \end{array} \\
(2) & 34 - \frac{5}{2\lambda}(\lambda^2 - \frac{1}{4}) \geq 0, \\
(3) & \begin{array}{l} \lambda\,(-5n_1 - n_1^2 - 5n_2 + 2n_1 n_2 - n_2^2 - 5n_3 + 2n_2 n_3 - n_3^2 - 5n_4 + 2n_1 n_4 \\ \quad -n_4^2 - 5n_5 + 2n_3 n_5 - n_5^2 - 5n_6 + 2n_4 n_6 + 2n_5 n_6 - n_6^2) = 3, \end{array} \\
(4) & \begin{array}{l} -n_1 + n_2 + n_4 \geq 0,\ n_1 - n_2 + n_3 \geq 0,\ n_2 - n_3 + n_5 \geq 0, \\ n_1 - n_4 + n_6 \geq 0,\ n_3 - n_5 + n_6 \geq 0,\ n_4 + n_5 - n_6 \geq 0, \end{array} \\
(5) & \begin{array}{l} -10 - b_5 + n_1 + n_5 \geq 0,\ -10 - b_6 + n_2 + n_6 \geq 0,\ -5 + b_5 - b_6 + n_3 - n_5 + n_6 \geq 0, \\ -5 - b_5 + b_6 + n_4 + n_5 - n_6 \geq 0,\ b_5 \geq 0,\ b_6 \geq 0. \end{array}
\end{array} \tag{79}$$

We can find all solution to (79) numerically by testing all lattice points in the polytope defined by the above inequalities. We find precisely 6930 solutions for $\lambda = \frac{3}{2}$ and 6990 solutions for $\lambda = \frac{1}{2}$, giving a total of 13,920 solutions. Presenting all these solutions is obviously un-illustrative. A few examples are the following. For

$$\lambda = \frac{1}{2}, \tag{80}$$

we find one solution to be

$$(n_1, n_2, n_3, n_4, n_5, n_6) = (0, 0, 5, 5, 10, 12). \tag{81}$$

This corresponds to the vacuum defined by

$$\eta = 15\ell - 5E_1 - 5E_2 - 3E_3 \ . \tag{82}$$

For

$$\lambda = \frac{3}{2}, \tag{83}$$

we find one solution to be

$$(n_1, n_2, n_3, n_4, n_5, n_6) = (2, 4, 8, 2, 10, 1), \tag{84}$$

corresponding to the vacuum

$$\eta = 22\ell - 10E_1 - 14E_2 - 17E_3 \ . \tag{85}$$

We will not present any further solutions for the higher del Pezzo surfaces because the exercise is not enlightening. In general, we see that because the inequality signs in constraints (1) and (5) in (47) run in opposite directions, the solutions will always be within some finite polytope. Additionally, the solutions are constrained to be special lattice points within the polytope which also obey (2), (3) and (4). In other words, for each generic base surface, there will always be a finite number of solutions. A very crude upper-bound to the number of solutions is, of course, the size of the polytope. For the del Pezzo surfaces, this is roughly 16^N, where N is the number of generators of the Mori cone in Table 1.

7 The Particle Spectrum in Heterotic Compactifications

We now address the key issue of this paper, namely, computing the particle spectra of grand unified theories in four dimensions arising from the compactification of heterotic theory on a Calabi-Yau threefold X endowed with a vector bundle V. We briefly review the requisite quantities in such a calculation.

Consider the $E_8 \times E_8$ gauge group of heterotic theory. In heterotic M-theory, one E_8 lives on the "observable" nine-brane while the other E_8 is restricted to the "hidden" brane. We will focus on the observable brane only and, hence, consider a single E_8 gauge group. Now compactify on a Calabi-Yau threefold X. A vector bundle V on X breaks this E_8 group down to some GUT group in the low energy theory. The canonical example is to take

$$V = TX \ , \tag{86}$$

where TX is the tangent bundle of X. See, for example, [17]. Since X has $SU(3)$ holonomy, it follows that V has the structure group $SU(3)$. This is known as the "standard embedding." The gauge connection on V is then identified with the spin connection of X. The low-energy GUT group is the commutant of $SU(3)$ in E_8, which is E_6. In other words, we have the breaking pattern

$$E_8 \to SU(3) \times E_6 \ . \tag{87}$$

The relevant fermionic fields in the low-energy four dimensional theory arise from the decomposition of the gauginos in the vector supermultiplet of the ten dimensional theory which transforms as the 248 of E_8. Under the decomposition (87), one finds that

$$248 \to (1,78) \oplus (3,27) \oplus (\overline{3},\overline{27}) \oplus (8,1) \ . \tag{88}$$

To be observable at low energy, the fermion fields transforming under the E_6 must be massless modes of the Dirac operator on X [17, 30]. It was shown in [30] that the number of massless modes for a given representation equals the dimension of a certain cohomology group. Let us first consider the representation $(1,78)$ in (88). In this case, we note that $h^1(X,\mathcal{O}_X)$ vanishes while

$$n_{78} = h^0(X,\mathcal{O}_X) = 1 \ . \tag{89}$$

These are the gauginos of a vector supermultiplet transforming in the 78 representation of E_6. For the other representations, the zeroth cohomology groups vanish and we have the following.

$$n_{27} = h^1(X,TX), \quad n_{\overline{27}} = h^1(X,TX^*), \tag{90}$$

and

$$n_1 = h^1(X,\mathrm{End(TX)}) = h^1(X,TX\otimes TX^*) \ . \tag{91}$$

Now

$$H^1(X,TX) \simeq H^{2,1}_{\bar{\partial}}(X),\ H^1(X,TX^*) \simeq H^{1,1}_{\bar{\partial}}(X), \tag{92}$$

where $H^{p,q}_{\bar{\partial}}(X)$ are the Dolbeault cohomology groups of X. It follows that

$$n_{27} = h^{2,1}, \quad n_{\overline{27}} = h^{1,1} \tag{93}$$

where $h^{1,1}$ and $h^{2,1}$ are the Betti numbers of the Calabi-Yau threefold X. Each such multiplet is the fermionic component of a chiral superfield transforming in the 27 or $\overline{27}$ representation of E_6. The remaining quantity, $h^1(X,TX\otimes TX^*)$, is a familiar object in deformation theory. It corresponds to the number of moduli of infinitesimal deformations of the tangent bundle

TX. Therefore, the complex scalar superpartners of the fermions transforming as singlets under E_6 are the bundle moduli of V. These form $n_1 = h^1(X, TX \otimes TX^*)$ chiral superfields, each an E_6 singlet. We remark that the 8 of $SU(3)$ is actually in the traceless part $Ad(TX)$ of $TX \otimes TX^*$. Note, however, that $TX \otimes TX^* = \mathcal{O}_X \oplus Ad(TX)$, where $\mathcal{O}_X$ is the trivial bundle on X, and that $h^1(X, \mathcal{O}_X)$ vanishes. Therefore $h^1(X, Ad(X)) = h^1(X, TX \otimes TX^*)$.

Since the Dolbeault cohomology groups for Calabi-Yau threefolds are known, one can compute the 27 and $\overline{27}$ part of the E_6 particle spectrum. Furthermore, the number of moduli of V can be found in a straight-forward manner. This has been discussed, for example, in [17]. It is important to note, however, that this can only be accomplished because one has chosen the standard embedding $V = TX$.

Having discussed the standard embedding, let us move on to so-called "non-standard" embeddings. It was realized in [5, 6, 30] that using such vector bundles one could get, in addition to E_6, more appealing GUT groups such as $SU(5)$ and $SO(10)$. This is done by taking V not to be the tangent bundle TX as was done above, but some more general holomorphic vector bundle V with structure group G. Since V is no longer TX, G need not be $SU(3)$. Then, the low-energy effective theory has gauge group H, where H is the commutant of G in E_8. For example, if we take V to be an $SU(4)$ bundle, then the low-energy GUT group is $SO(10)$. If V has structure group $SU(5)$, the low-energy GUT group is $SU(5)$. The decomposition of the 248 of E_8 under these groups is as follows.

$E_8 \to G \times H$	
$SU(3) \times E_6$	$248 \;\; \to \;\; (1,78) \oplus (3,27) \oplus (\overline{3},\overline{27}) \oplus (8,1)$
$SU(4) \times SO(10)$	$248 \;\; \to \;\; (1,45) \oplus (4,16) \oplus (\overline{4},\overline{16}) \oplus (6,10) \oplus (15,1)$
$SU(5) \times SU(5)$	$248 \;\; \to \;\; (1,24) \oplus (5,\overline{10}) \oplus (\overline{5},10) \oplus (10,5) \oplus (\overline{10},\overline{5}) \oplus (24,1)$

(94)

For a non-standard vector bundle V, the zero mode spectrum continues to depend on the dimensions of certain cohomology groups. In this paper, for specificity, we will be primarily interested in $SU(5)$ GUTS. To differentiate the two $SU(5)$ groups in $SU(5) \times SU(5)$, denote the structure group of V, the first factor, by $SU(5)_G$ and the low energy GUT group, the second factor, by $SU(5)_H$. From (94), we see that the 1 , 5, $\overline{5}$, 10, $\overline{10}$ and 24 representations of $SU(5)_G$ are paired with the 24 ,$\overline{10}$, 10, 5, $\overline{5}$, and 1 respectively of $SU(5)_H$. Furthermore, note that the 5, $\overline{5}$, 10, $\overline{10}$ and 24 representations of $SU(5)_G$ are associated with the vector bundles V, V^*, $\wedge^2 V$, $\wedge^2 V^*$ and $V \otimes V^*$. Then, the spectrum of zero mass fields transforming as the 24 ,$\overline{10}$, 10, 5, $\overline{5}$, and 1 of $SU(5)_H$ are the following. First, as previously,

$$n_{24} = 1 \ , \tag{95}$$

indicating that there is a single vector supermultiplet transforming in the adjoint 24 representation of $SU(5)_H$. The remaining representations all occur as chiral superfields. Their spectrum is given by

$$n_{\overline{10}} = h^1(X,V), \quad n_{10} = h^1(X,V^*),$$
$$n_5 = h^1(X,\wedge^2 V), \quad n_{\bar{5}} = h^1(X,\wedge^2 V^*) \tag{96}$$

and

$$n_1 = h^1(X, V \otimes V^*) \ . \tag{97}$$

This is a straightforward generalization of the formula for the spectrum in the standard embedding case. Now, however, these cohomology groups are unrelated to the Dolbeault cohomology of X and, hence, far more difficult to calculate. It will be the task of the remainder of this paper to present a general method for computing the quantities in (96) and (97). Finally, note that even though we will work within the context of $SU(5)$ GUTs, our formalism and results generalize in a straight-forward manner to any vector bundle V.

7.1 Constraints from the Index Theorem

Before computing the cohomology groups in (96) and (97), let us see what simplifications can be achieved using the index theorem. First, we apply Serre duality. This states that

$$H^i(X,V) \simeq H^{3-i}(X, V^* \otimes K_X) \simeq H^{3-i}(X,V^*), \tag{98}$$

where we have used the fact that K_X is trivial on a Calabi-Yau threefold X. Therefore, we have

$$H^0(X,V) \simeq H^3(X,V^*), \quad H^0(X,V^*) \simeq H^3(X,V) \tag{99}$$

and

$$H^1(X,V) \simeq H^2(X,V^*), \quad H^1(X,V^*) \simeq H^2(X,V). \tag{100}$$

It can be shown that for a stable vector bundle V of rank greater than one and with vanishing first Chern class,

$$H^0(X,V) = H^0(X,V^*) = 0 \ . \tag{101}$$

It then follows from (99) that

$$H^3(X,V^*) = H^3(X,V) = 0 \ . \tag{102}$$

The Atiyah-Singer index theorem states that

$$\sum_{i=0}^{3}(-1)^i h^i(X,V) = \int_X \mathrm{ch}(V)\mathrm{td}(X) = \frac{1}{2}\int_X c_3(V), \tag{103}$$

where, in deriving the final term, we have used the facts from (5) and (33) that $c_1(TX)$ and $c_1(V)$ both vanish. Expressions (99), (100), (101) and (102) allow us to simplify (103) to

$$-h^1(X,V) + h^1(X,V^*) = \frac{1}{2}\int_X c_3(V). \tag{104}$$

Now, recall the physical condition (39) that our theory have three quark/lepton generations. It then follows from (38) that

$$\frac{1}{2}\int_X c_3(V) = 3. \tag{105}$$

Therefore, the index theorem becomes

$$-h^1(X,V) + h^1(X,V^*) = 3. \tag{106}$$

This will be an important constraint for us. It implies that, of the two terms $h^1(X,V)$ and $h^1(X,V^*)$ that we need to compute the spectrum, it suffices to calculate only one of them and the other will differ from it by ± 3.

Similarly, we have that

$$H^0(X,\wedge^2 V) = H^0(X,\wedge^2 V^*) = 0, \tag{107}$$

since $\wedge^2 V$ and $\wedge^2 V^*$ are both stable bundles and have vanishing first Chern class. Therefore, the above arguments lead to the index theorem

$$-h^1(X,\wedge^2 V) + h^1(X,\wedge^2 V^*) = \frac{1}{2}\int_X c_3(\wedge^2 V)\ . \tag{108}$$

We have computed the Chern classes of the antisymmetrized products of V in Appendix B. Using (33), (339) and the fact that we have chosen the structure group of V to be $SU(5)$, it follows that

$$c_3(\wedge^2 V) = c_3(V). \tag{109}$$

Therefore, the physical constraint (105) simplifies the index relation (108) to

$$-h^1(X,\wedge^2 V) + h^1(X,\wedge^2 V^*) = 3\ . \tag{110}$$

As above, this constraint says that we need to compute only one of $h^1(X,\wedge^2 V)$ and $h^1(X,\wedge^2 V^*)$. The other will be determined by adding or subtracting 3 according to (110).

Finally, the above index theorem is inert when applied to $V \otimes V^*$. This is because Serre duality (98) implies

$$H^0(X, V \otimes V^*) \simeq H^3(X, (V \otimes V^*)^*) = H^3(X, V \otimes V^*) \tag{111}$$

and

$$H^1(X, V \otimes V^*) \simeq H^2(X, V \otimes V^*). \tag{112}$$

Therefore, application of the index theorem gives

$$0 = \frac{1}{2} \int_X c_3(V \otimes V^*). \tag{113}$$

This is consistent with the fact that

$$c_3(V \otimes V^*) = 0, \tag{114}$$

which holds since for any vector bundle W, $c_3(W^*) = -c_3(W)$, and for $W = V \otimes V^*$ we have $W = W^*$. Hence, we must compute $h^1(X, V \otimes V^*)$ directly.

7.2 Determining The Spectral Data

It follows from the previous discussion that the vector bundles over X that we will need to determine the spectrum are of five types

$$U = V,\ V^*,\ \wedge^2 V,\ \wedge^2 V^*,\ V \otimes V^* \ . \tag{115}$$

For the first four bundles, it will be necessary to extract the relevant cohomological data using the associated spectral data. The relevant quantity for the bundle $V \otimes V^*$, namely $h^1(X, V \otimes V^*)$, can be computed using a different technique and will be discussed separately in a later section.

Recall, from the discussion in Section 4.1, that the Fourier-Mukai transformation relates U to its spectral data as

$$(\mathcal{C}_U, \mathcal{N}_U) \stackrel{FM}{\longleftrightarrow} U \tag{116}$$

where $\mathcal{C}_U$ and $\mathcal{N}_U$ are a divisor on X and a line bundle on $\mathcal{C}_U$ respectively. As we will see, for $U = V,\ V^*$ the line bundle $\mathcal{N}_U$ on $\mathcal{C}_U$ is the restriction of a line bundle on X. We will extend the notation discussed previously and not distinguish between $\mathcal{N}_U$ on X and $\mathcal{N}_U$ restricted to $\mathcal{C}_U$, denoting both as $\mathcal{N}_U$. In fact, to simplify our notation, if L is any line bundle on X we will denote both L and $L|_{\mathcal{C}_U}$ as L. Note, however, that not all line bundles on $\mathcal{C}_U$ are restrictions of line bundles on X. Specifically, we will show that this is the case for $U = \wedge^2 V,\ \wedge^2 V^*$. We turn, therefore, to computing the spectral data for each of $U = V,\ V^*,\ \wedge^2 V,\ \wedge^2 V^*$. Before computing, we note that the curve defined by the intersection of the spectral cover with the zero section, that is,

$$\bar{c}_U = \mathcal{C}_U \cap \sigma \ , \tag{117}$$

will play an important role in our analysis. We will refer to this as the "support curve." Note that in (117) we use the actual intersection "$\cap$", as opposed to the intersection "$\cdot$" in cohomology, since $\mathcal{C}_U$ and σ are specific surfaces. Henceforth, writing $\mathcal{C}_U \cdot \sigma$, for example, will refer to the intersection of the class $[\mathcal{C}_U]$ with the class $[\sigma]$. As defined, $\bar{c}_U$ is a curve in σ, $\mathcal{C}_U$ and X. Now, consider $\pi_\sigma(\bar{c}_U)$ in B. Since $\pi_\sigma : \sigma \to B$ is an isomorphism we will, henceforth, not distinguish between $\pi_\sigma(\bar{c}_U)$ and $\bar{c}_U$, denoting them both by $\bar{c}_U$.

7.2.1 Spectral Data for V

The spectral data for $U = V$ was presented in Section 4. For the readers' convenience, we repeat those expressions here. The spectral cover for a rank n vector bundle V has the form

$$\mathcal{C}_V \in |n\sigma + \pi^*\eta| \ , \tag{118}$$

where η is some effective curve class in B. The spectral line bundle $\mathcal{N}_V$ is specified by expression (24)

$$c_1(\mathcal{N}_V) = n(\frac{1}{2} + \lambda)\sigma + (\frac{1}{2} - \lambda)\pi^*\eta + (\frac{1}{2} + n\lambda)\pi^* c_1(TB) \ , \tag{119}$$

where λ is a rational number satisfying conditions (25).

7.2.2 Spectral Data for V^*

Let us now proceed to determine the necessary data for $U = V^*$. We first recall from (4) that there is a natural involution τ on X. The spectral cover $\mathcal{C}_{V^*}$ is given by $\tau\mathcal{C}_V$. Since the linear system $|\mathcal{C}_V|$ is invariant under τ, we have that $|\mathcal{C}_{V^*}| = |\mathcal{C}_V|$. Now, note that the Chern classes have the property that

$$c_i(V^*) = (-1)^i c_i(V) \tag{120}$$

for any vector bundle V. Using this relation and expressions (33) we can compute the Chern classes V^*. We see that, since $|\mathcal{C}_{V^*}| = |\mathcal{C}_V|$, these Chern classes will be consistent with choosing $\mathcal{N}_{V^*}$ to be $\mathcal{N}_V$ in (24) with

$$\lambda \to -\lambda \ . \tag{121}$$

In summary, we have

$$
\begin{aligned}
|\mathcal{C}_{V^*}| &= |\mathcal{C}_V| \\
c_1(\mathcal{N}_{V^*}) &= n(\frac{1}{2}-\lambda)\sigma + (\frac{1}{2}+\lambda)\pi^*\eta + (\frac{1}{2}-n\lambda)\pi^*c_1(TB)\ .
\end{aligned} \tag{122}
$$

7.2.3 Spectral Data for $\wedge^2 V$

Next, let us construct the spectral data for $U = \wedge^2 V$. The linear system of the spectral cover is easily determined, again, by considering the Chern classes. In Appendix B, we compute the Chern classes of the antisymmetric products of a vector bundle. The result in (339) confirms that the vector bundle $\wedge^2 V$ must have rank $\frac{1}{2}n(n-1)$, as expected from antisymmetry. Furthermore, since for our vector bundle $c_1(V) = 0$, it follows from (339) that

$$
c_2(\wedge^2 V) = (n-2)c_2(V), \tag{123}
$$

where $c_2(V)$ is given in (33). We can now try to recast $c_2(\wedge^2 V)$ in the same form as (33), but with n replaced by $\frac{1}{2}n(n-1)$ and possibly new values for η and λ. We see that the first term in $c_2(\wedge^2 V)$, the horizontal component of the Chern class, can be put in the same form as (33) by choosing a new base curve η' defined as $\eta' = (n-2)\eta$. Since one can show that this horizontal term only depends on the spectral cover, we can conclude that the spectral cover for $\wedge^2 V$ is given by

$$
\mathcal{C}_{\wedge^2 V} \in \left| \frac{n(n-1)}{2}\sigma + (n-2)\pi^*\eta \right| \ . \tag{124}
$$

However, the remaining terms in $c_2(\wedge^2 V)$ do not arise from any line bundle on X of the form specified in (24). It follows that $\mathcal{N}_{\wedge^2 V}$ is a line bundle on $\mathcal{C}_{\wedge^2 V}$ which is not the restriction of any bundle on X. That is, $c_1(\mathcal{N}_{\wedge^2 V})$ is represented by some curve in $\mathcal{C}_{\wedge^2 V}$, but this curve is not the complete intersection of $\mathcal{C}_{\wedge^2 V}$ with any divisor in X [2]. What then is $c_1(\mathcal{N}_{\wedge^2 V})$? This is a daunting problem. Happily, as we will show in the next section, to compute the cohomology of $\wedge^2 V$ it will be necessary to find not all of $\mathcal{N}_{\wedge^2 V}$ but, rather, its restriction $\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}}$, where $\overline{c}_{\wedge^2 V}$ is the support curve defined by

$$
\overline{c}_{\wedge^2 V} = \mathcal{C}_{\wedge^2 V} \cap \sigma. \tag{125}
$$

This we can accomplish, as we now show.

[2] A theorem of Lefschetz shows that any curve class on $\mathcal{C}_V$ comes from a divisor in X, when $\mathcal{C}_V$ is smooth and very ample. But $\mathcal{C}_{\wedge^2 V}$ has no reason to satisfy these conditions, so the Lefschetz theorem does not apply.

Consider the action of the involution τ, defined in (4), on the spectral cover surface $\mathcal{C}_V$. Denote the transformed surface by $\tau\mathcal{C}_V$ and intersect it with $\mathcal{C}_V$ to obtain the curve $\tau\mathcal{C}_V \cap \mathcal{C}_V$. It is not hard to show that this curve, which has multiple components, decomposes as

$$\tau\mathcal{C}_V \cap \mathcal{C}_V = (\mathcal{C}_V \cap \sigma) \cup (\mathcal{C}_V \cap \sigma_2) \cup D \ , \tag{126}$$

where σ, we recall, is the zero section and σ_2 is the trisection that intersects each fiber at the three non-trivial points of order 2. It is given by

$$\sigma_2 \in |3\sigma + 3\pi^*(c_1(TB))|. \tag{127}$$

Note, however, that there is a third component curve in (126) which we denote by D. Now, the linear system $|\mathcal{C}_V|$ is invariant under τ. Therefore, using expressions (23) and (127), we can solve for D. We find that it is a representative of the class

$$D \in \left[\sigma \cdot \pi^*((2n-4)\eta - (n^2-n)c_1(TB)) + (\eta^2 - 3\eta \cdot c_1(TB))F\right] \ . \tag{128}$$

Note that D is contained in X. Let $\overline{c}_{\wedge^2 V}$ be the curve in the base associated with $\overline{c}_{\wedge^2 V}$ defined in (125). We remind the reader that, notationally, we are not distinguishing these curves. Then, one can show that D is actually the double cover of the support curve $\overline{c}_{\wedge^2 V}$, with covering map $\pi_D : D \to \overline{c}_{\wedge^2 V}$. There are, in principle, a number of branch points and the associated ramification points of this mapping. The branch divisor in $\overline{c}_{\wedge^2 V}$ will be denoted by Br, whereas the ramification divisor in D is written as R. The numbers of branch points and ramification points are given by $\deg(Br)$ and $\deg(R)$ respectively. Note that

$$\deg(Br) = \deg(R) \ . \tag{129}$$

In the following, we may, for simplicity, denote both the divisor and its degree by the same symbol. For example, we will write R for both the ramification divisor and its degree. This divisor can be obtained as the intersection of D with the zero section and the points of order two. That is,

$$R = D \cap (\sigma + \sigma_2) \ . \tag{130}$$

Numerically, R can be computed using

$$R = D \cdot (4\sigma + 3\pi^*(c_1(TB)) \ , \tag{131}$$

where D is given in (128). While the integer R is always defined by formula (131), the divisor R is not always determined by formula (130) unless the intersection is proper. In a crucial situation which we will encounter, a component of D is actually contained in σ. In this situation, the divisor cannot be uniquely determined. To our rescue comes intersection theory [26], which says that the line bundle $\mathcal{O}_D(R)$ is nevertheless well defined. It is given by a refinement of (131), namely, $\mathcal{O}_D(R)$ is the restriction to D of the line bundle $\mathcal{O}_X(R) = \mathcal{O}_X(4\sigma + 3\pi^* c_1(TB))$ on X. That is,

$$\mathcal{O}_D(R) = \mathcal{O}_X(4\sigma + 3\pi^* c_1(TB))|_D \ . \tag{132}$$

The definition of D as a double cover of $\overline{c}_{\wedge^2 V}$ allows us to relate the spectral line bundle $\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}}$ to $\mathcal{N}_V^{\otimes 2}|_D$. After a detailed analysis, we can prove that

$$\pi_D^* \mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}} = \mathcal{N}_V^{\otimes 2}|_D \otimes \mathcal{O}_D(-R), \tag{133}$$

or, equivalently,

$$\mathcal{N}_V^{\otimes 2}|_D = \pi_D^* \mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}} \otimes \mathcal{O}_D(R) \tag{134}$$

where $\mathcal{O}_D(R)$ is given in (132). Here and henceforth, we use the following notation. As discussed earlier, $\mathcal{N}_U$ denotes a line bundle both on X and restricted to $\mathcal{C}_U$. In either case $\mathcal{N}_U|_{\bar{c}_U}$ is well-defined. Now consider $\pi_\sigma|_{\bar{c}_U *}(\mathcal{N}_U|_{\bar{c}_U})$, which is a line bundle on the curve $\bar{c}_U$ in the base B. Since $\pi_\sigma : \sigma \to B$ is an isomorphism, we will not distinguish between $\pi_\sigma|_{\bar{c}_U *}(\mathcal{N}_U|_{\bar{c}_U})$ and $\mathcal{N}_U|_{\bar{c}_U}$, denoting them both by $\mathcal{N}_U|_{\bar{c}_U}$. It then follows that

$$c_1(\mathcal{N}_V^{\otimes 2}|_D) = 2 c_1(\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}}) + R, \tag{135}$$

where we have used the fact that π_D is of degree two and that $c_1(\mathcal{O}_D(R)) = R$. Now, note that

$$c_1(\mathcal{N}_V^{\otimes 2}|_D) = 2 c_1(\mathcal{N}_V) \cdot D, \tag{136}$$

where $c_1(\mathcal{N}_V)$ is given in (24). Combining (135) and (136) yields the desired result

$$c_1(\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}}) = c_1(\mathcal{N}_V) \cdot D - \frac{R}{2}. \tag{137}$$

In summary, from (124) and (137), we conclude that the required part of the spectral data for $\wedge^2 V$ is given by

$$\begin{aligned} \mathcal{C}_{\wedge^2 V} &\in \left| \frac{n(n-1)}{2}\sigma + (n-2)\pi^*\eta \right| , \\ c_1(\mathcal{N}_{\wedge^2 V})|_{\overline{c}_{\wedge^2 V}} &= \left(n(\frac{1}{2} + \lambda)\sigma + (\frac{1}{2} - \lambda)\pi^*\eta + (\frac{1}{2} + n\lambda)\pi^* c_1(TB) \right) \cdot D - \frac{R}{2} \ , \end{aligned} \tag{138}$$

with D and R are defined in (128) and (131) respectively.

We can be even more specific about the structure of $\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}}$. Pushing equation (134) down onto $\overline{c}_{\wedge^2 V}$, one finds that

$$\pi_{D*}(\mathcal{N}_V^{\otimes 2}|_D) = \mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}} \otimes \pi_{D*}(\mathcal{O}_D(R)) \ . \tag{139}$$

However, $\mathcal{O}_D(R)$ pushed down onto $\overline{c}_{\wedge^2 V}$ is a rank two vector bundle which is the direct sum of two line bundles,

$$\pi_{D*}(\mathcal{O}_D(R)) = \mathcal{O}_{\overline{c}_{\wedge^2 V}} \oplus \mathcal{O}_{\overline{c}_{\wedge^2 V}}(\frac{Br}{2}) \ , \tag{140}$$

where $Br = \pi_D(R)$ is the divisor of branch points in $\overline{c}_{\wedge^2 V}$. Substituting (140) into (139) gives

$$\pi_{D*}(\mathcal{N}_V^{\otimes 2}|_D) = \mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}} \oplus \left(\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}} \otimes \mathcal{O}_{\overline{c}_{\wedge^2 V}}(\frac{Br}{2})\right) \ . \tag{141}$$

This implicitly contains an expression for $\mathcal{N}_{\wedge^2 V}|_{\overline{c}_{\wedge^2 V}}$, not just its first Chern class, that we will find useful later in the paper.

7.2.4 Spectral Data for $\wedge^2 V^*$

First, using the fact that

$$\wedge^2 V^* = (\wedge^2 V)^* \ , \tag{142}$$

we conclude that

$$|\mathcal{C}_{\wedge^2 V^*}| = |\mathcal{C}_{\wedge^2 V}|. \tag{143}$$

Furthermore, we know from (120) and (121) that the Chern classes of V^* are obtained from those of V by letting $\lambda \to -\lambda$. Using this and expression (138) we can compute $c_1(\mathcal{N}_{\wedge^2 V^*})|_{\overline{c}_{\wedge^2 V^*}}$. In summary, we find, using (125) and (138), that

$$\begin{aligned} \mathcal{C}_{\wedge^2 V^*} &\in \left|\frac{n(n-1)}{2}\sigma + (n-2)\pi^*\eta\right| \ , \\ c_1(\mathcal{N}_{\wedge^2 V^*})|_{\overline{c}_{\wedge^2 V^*}} &= \left(n(\frac{1}{2}-\lambda)\sigma + (\frac{1}{2}+\lambda)\pi^*\eta + (\frac{1}{2}-n\lambda)\pi^* c_1(TB)\right)\cdot D - \frac{R}{2} \ . \end{aligned} \tag{144}$$

Similarly, for the bundle $\wedge^2 V^*$, expression (141) becomes

$$\pi_{D*}(\mathcal{N}_{V^*}^{\otimes 2}|_D) = \mathcal{N}_{\wedge^2 V^*}|_{\overline{c}_{\wedge^2 V^*}} \oplus \left(\mathcal{N}_{\wedge^2 V^*}|_{\overline{c}_{\wedge^2 V^*}} \otimes \mathcal{O}_{\overline{c}_{\wedge^2 V^*}}(\frac{Br}{2})\right) \ . \tag{145}$$

where Br is the divisor of branch points in $\overline{c}_{\wedge^2 V^*}$. It is not hard to show that $\overline{c}_{\wedge^2 V^*} = \overline{c}_{\wedge^2 V}$.

7.3 Computing the Particle Spectrum

We know from the discussions at the beginning of this section that to determine the particle spectrum one must compute $H^1(X,U)$ for $U = V, V^*, \wedge^2 V, \wedge^2 V^*$ and $V \otimes V^*$. Now, recall that any such U and X have the following structure

$$\begin{array}{ccc} U & & \\ \downarrow & & \\ X & \stackrel{\pi}{\to} & B \;. \end{array} \tag{146}$$

The standard technique for finding $H^1(X,U)$ on such a structure is to evoke the Leray spectral sequence. This reduces the cohomology of U over X to that of derived functors $R^i\pi_* U$ over the base B. Since the fibers of the projection map $\pi : X \to B$ are one-dimensional, we see that for any vector bundle U on X, the Leray spectral sequence reduces to a single long exact sequence

$$0 \to H^1(B, \pi_* U) \to H^1(X,U) \to H^0(B, R^1\pi_* U) \to H^2(B, \pi_* U) \to \ldots \tag{147}$$

where $R^i\pi_*$ is the i-th right derived functor for the push-forward map π_*. The reader is referred to [22, 23] for a discussion of Leray sequences. We first recall some key facts from [31] concerning the properties of $\pi_* V$ and $R^1\pi_* V$. On the base surface B of our elliptic fibration, we have

$$\pi_* V = 0, \quad \mathrm{rk}_B(R^1\pi_* V) = 0 \;. \tag{148}$$

This follows from the fact that V is a vector bundle corresponding to an irreducible spectral cover. Similarly, this holds for $U = V^*, \wedge^2 V$ and $\wedge^2 V^*$. That is

$$\pi_* U = 0, \quad \mathrm{rk}_B(R^1\pi_* U) = 0 \tag{149}$$

for $U = V, V^*, \wedge^2 V, \wedge^2 V^*$. For $U = V \otimes V^*$, however, it is not true that $\pi_* U$ vanishes. For this reason, as mentioned earlier, we will compute $H^1(X, V \otimes V^*)$ separately, using a different formalism. Here, we will restrict U to be $V, V^*, \wedge^2 V, \wedge^2 V^*$ only.

It follows from the first equation in (149) that

$$H^1(B, \pi_* U) = H^2(B, \pi_* U) = 0. \tag{150}$$

The sequence (147) then implies

$$H^1(X,U) \simeq H^0(B, R^1\pi_* U). \tag{151}$$

The first direct image $R^1\pi_*U$ does not vanish identically on B but, rather, is a torsion sheaf. It is supported on the curve

$$\bar{c}_U = \mathcal{C}_U \cap \sigma \tag{152}$$

in B. The genus of $\bar{c}_U$ can be obtained from the adjunction formula

$$2g - 2 = \bar{c}_U \cdot (\bar{c}_U + c_1(K_B)) \ . \tag{153}$$

To recapitulate, our spectrum calculation simplifies to finding the global holomorphic sections of $R^1\pi_*U$ on the support curve $\bar{c}_U \subset B$. That is,

$$H^1(X, U) \simeq H^0(\bar{c}_U, R^1\pi_*U|_{\bar{c}_U}) \ . \tag{154}$$

It is important to note, however, that even though π_*U and $\text{rk}_B(R^1\pi_*U)$ vanish, $R^1\pi_*U$ need not be zero.

7.4 The First Image $R^1\pi_*U$

Let us determine, using the Fourier-Mukai techniques presented in Section 4.1, the torsion sheaf $R^1\pi_*U$. The commutativity of the diagram in (32) allows us to write

$$\pi' \circ \pi_1 = \pi_{\mathcal{C}} \circ \pi_2 \ . \tag{155}$$

This implies that, in the derived category, the functors R of these projection maps obey

$$R\pi'_* \circ R\pi_{1*} = R\pi_{\mathcal{C}*} \circ R\pi_{2*} \ . \tag{156}$$

However, π_1 and $\pi_{\mathcal{C}}$ are finite covering maps so their higher direct images vanish. This means that

$$R\pi_{1*} = \pi_{1*}, \quad R\pi_{\mathcal{C}*} = \pi_{\mathcal{C}*} \ . \tag{157}$$

Subsequently, (156) becomes

$$R^i\pi'_* \circ \pi_{1*} = \pi_{\mathcal{C}*} \circ R^i\pi_{2*} \ , \quad i \geq 0. \tag{158}$$

Applying the $i = 1$ case of (158) to the sheaf $\pi_2^*\mathcal{N}_U \otimes \mathcal{P}$, we have that

$$R^1\pi'_* \circ \pi_{1*}(\pi_2^*\mathcal{N}_U \otimes \mathcal{P}) = \pi_{\mathcal{C}*} \circ R^1\pi_{2*}(\pi_2^*\mathcal{N}_U \otimes \mathcal{P}) \ . \tag{159}$$

Now we recall the definition of U from (31) and use the projection formula

$$R^1\pi_{2*}(\pi_2^*\mathcal{N}_U \otimes \mathcal{P}) = \mathcal{N}_U \otimes R^1\pi_{2*}\mathcal{P}. \tag{160}$$

Then (159) simplifies to

$$R^1\pi'_*U = \pi_{\mathcal{C}*}(\mathcal{N}_U \otimes R^1\pi_{2*}\mathcal{P}) \ . \tag{161}$$

From now on, using the fact that $X \simeq X'$, we will omit the prime and replace π' by π. The left hand side is precisely the term we desire, while the right hand side can be further simplified using the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}_U \times_B X' & \xrightarrow{i_{\mathcal{C}\times_B X'}} & X \times_B X' \leftarrow \mathcal{P} \\ \downarrow \pi_2 & & \downarrow \pi_X \\ \mathcal{C}_U & \xrightarrow[i_{\mathcal{C}}]{} & X \end{array} \ . \tag{162}$$

Therefore,

$$R^1\pi_{2*}\mathcal{P} = i^*_{\mathcal{C}}(R^1\pi_{X*}\mathcal{P}) \ . \tag{163}$$

Finally, we know that

$$R^1\pi_{X*}\mathcal{P} = \sigma_* K_B \ , \tag{164}$$

and thus

$$R^1\pi_{2*}\mathcal{P} = i^*_{\mathcal{C}}\sigma_* K_B \ , \tag{165}$$

where $\sigma : B \to X$ is the zero section map discussed in Section 2. Substituting this into (161) gives us

$$R^1\pi_*U = \pi_{\mathcal{C}*}(\mathcal{N}_U \otimes (i^*_{\mathcal{C}}\sigma_* K_B)) \ . \tag{166}$$

It is clear that $i^*_{\mathcal{C}}\sigma_* K_B$ is a sheaf on $\mathcal{C}_U$ with support on the curve $\bar{c}_U = \mathcal{C}_U \cap \sigma$. Note that this can be thought of as $\pi^*_{\mathcal{C}} K_B|_{\bar{c}_U}$. Then, for any line bundle $\mathcal{N}_U$, we can identify

$$\mathcal{N}_U \otimes (i^*_{\mathcal{C}}\sigma_* K_B) = (\mathcal{N}_U \otimes \pi^*_{\mathcal{C}} K_B)|_{\bar{c}_U} \ . \tag{167}$$

For $U = V,\ V^*$, when $\mathcal{N}_U$ is defined globally on X, we can replace $\pi^*_{\mathcal{C}}$ by π^* in (167) and write

$$\mathcal{N}_U \otimes (i^*_{\mathcal{C}}\sigma_* K_B) = (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U} \ . \tag{168}$$

Note that this latter equation does not apply to $U = \wedge^2 V,\ \wedge^2 V^*$ since, in these cases, $\mathcal{N}_U$ is defined on $\mathcal{C}_U$ only. Be that as it may, to avoid having to give separate discussions we will, henceforth, always denote this sheaf by $(\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}$. Whether the lift is by π^* or $\pi^*_{\mathcal{C}}$ will be clear from the context. Since, as discussed previously, we will not distinguish a line bundle on $\bar{c}_U \subset \mathcal{C}_U$ from the associated line bundle on $\bar{c}_U \subset B$, we can write

$$\pi_{\mathcal{C}*}(\mathcal{N}_U \otimes (i^*_{\mathcal{C}}\sigma_* K_B)) = (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U} \ . \tag{169}$$

Combining (166) and (169) with expression (154), we have

$$H^1(X,U) \simeq H^0(\bar{c}_U, (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}). \tag{170}$$

Since we know $\bar{c}_U$, $\mathcal{N}_U|_{\bar{c}_U}$ and $K_B|_{\bar{c}_U}$, this expression allows us, in principle, to compute $H^1(X,U)$ for $U = V, V^*, \wedge^2 V, \wedge^2 V^*$. In practice, this calculation will depend on the properties of the support curve $\bar{c}_U$.

7.5 Riemann-Roch Theorem on a Smooth Support Curve

Generically, the support curve $\bar{c}_U$ will be smooth and irreducible. However, our three-family constraint may make it reducible and even non-reduced and, therefore, much more difficult to analyze. Even in these cases, however, $\bar{c}_U$ may contain a smooth, irreducible component. It is of interest, therefore, to discuss the Riemann-Roch theorem for smooth curves. As we will see, this theorem is very helpful in computing either all, or part, of $H^0(\bar{c}_U, (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U})$.

The Riemann-Roch theorem states that for a smooth curve C and a line bundle $\mathcal{F}$ on C, we have

$$h^0(C,\mathcal{F}) - h^1(C,\mathcal{F}) = \deg(\mathcal{F}) - g(C) + 1, \tag{171}$$

where $g(C)$ is the genus of the curve and

$$\deg(\mathcal{F}) = c_1(\mathcal{F}) \tag{172}$$

is the degree of $\mathcal{F}$. In our problem, we need to calculate the term $h^0(C,\mathcal{F})$. If $\deg(\mathcal{F}) < 0$, the term $h^0(C,\mathcal{F})$ simply vanishes because there are no global holomorphic sections to a line bundle of negative degree. Now, we assume that $\deg(\mathcal{F}) \geq 0$. Using Serre duality, we have

$$h^1(C,\mathcal{F}) = h^0(C,\mathcal{F}^* \otimes K_C), \tag{173}$$

where K_C is the canonical bundle of C. Then, (171) becomes

$$h^0(C,\mathcal{F}) - h^0(C,\mathcal{F}^* \otimes K_C) = \deg(\mathcal{F}) - g(C) + 1. \tag{174}$$

Now,

$$\deg(\mathcal{F}^* \otimes K_C) = -\deg(\mathcal{F}) + \deg(K_C) = -\deg(\mathcal{F}) + 2g(C) - 2, \tag{175}$$

where we have used the fact that

$$\deg(K_C) = 2g(C) - 2. \tag{176}$$

When $\deg(\mathcal{F}^* \otimes K_C) < 0$, $h^0(C, \mathcal{F}^* \otimes K_C)$ vanishes and (174) becomes

$$h^0(C, \mathcal{F}) = \deg(\mathcal{F}) - g(C) + 1. \tag{177}$$

Thus, for line bundles $\mathcal{F}$ on C for which $\deg(\mathcal{F}^* \otimes K_C) < 0$, the so-called "stable range," we can compute $h^0(C, \mathcal{F})$ explicitly using (177). Note, however, that outside this range the Riemann-Roch theorem is not sufficient in determining $h^0(C, \mathcal{F})$.

As a simple example, let us assume that the support curve $\bar{c}_U$ is smooth and consider the line bundle $(\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}$. That is, take $C = \bar{c}_U$ and $\mathcal{F} = (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}$. Let us denote

$$d = \deg((\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}) = \int_{\bar{c}_U} c_1(\mathcal{N}_U \otimes \pi^* K_B). \tag{178}$$

If $d < 0$, then $h^0(\bar{c}_U, (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U})$ vanishes. Now assume that $d \geq 0$. It follows from (175) that

$$\deg((\mathcal{N}_U \otimes \pi^* K_B)|^*_{\bar{c}_U} \otimes K_{\bar{c}_U}) = -d + 2g(\bar{c}_U) - 2\ . \tag{179}$$

If expression (179) is non-negative, the Riemann-Roch theorem is insufficient to determine $h^0(\bar{c}_U, (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U})$. However, if

$$-d + 2g(\bar{c}_U) - 2 < 0, \tag{180}$$

that is, if $(\mathcal{N}_U \otimes \pi^* K_B)^* \otimes K_C$ is in the stable range, (177) implies that

$$h^0(\bar{c}_U, (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}) = d - g(\bar{c}_U) + 1\ . \tag{181}$$

To complete this computation, note that

$$c_1((\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}) = c_1(\mathcal{N}_U|_{\bar{c}_U}) + c_1(K_B|_{\bar{c}_U}). \tag{182}$$

Then, expression (181) becomes

$$h^0(\bar{c}_U, (\mathcal{N}_U \otimes \pi^* K_B)|_{\bar{c}_U}) = (-c_1(TB) + c_1(\mathcal{N}_U)) \cdot \bar{c}_U - g(\bar{c}_U) + 1. \tag{183}$$

Recalling that the genus $g(\bar{c}_U)$ can be computed by adjunction (153), and using (170), the above expression simplifies to

$$h^1(X, U) = \left(c_1(\mathcal{N}_U) - \frac{1}{2} c_1(TB) - \frac{1}{2} \bar{c}_U \right) \cdot \bar{c}_U\ . \tag{184}$$

Note that the first intersection is to be carried out in X where the remaining two intersections occur in the base B. This result is the final expression for $h^1(X, U)$ in the simple case that the support curve $\bar{c}_U$ is smooth and the degree of $\mathcal{N}_U \otimes \pi^* K_B$ falls in the stable range (180). Unfortunately, as we will see below, this simplified example is not realised in the realistic three-family constrained models of interest. Be that as it may, the general analysis of the Riemann-Roch theorem presented here will play an important role in determining part of $h^1(X, U)$ on smooth components of $\bar{c}_U$.

8 An Explicit Calculation

The calculation of $H^1(X,U)$ for $U = V, V^*, \wedge^2 V$ and $\wedge^2 V^*$ on the complicated curves $\bar{c}_U$ that are encountered in phenomenologically realistic theories is rather intricate. We find it expedient, and more enlightening, to present an explicit example that will illustrate all of the techniques necessary to compute $H^1(X,U)$. These techniques can be applied to most other examples that could arise. With this in mind, we choose the following example from the $SU(5)$ GUT vacua classified in Section 6. Recall from (64) and (66) that

$$B = \mathbb{F}_1,\ n = 5,\ \eta = 12S + 15\mathcal{E},\ \lambda = \frac{1}{2} \tag{185}$$

is an explicit solution which satisfies all of the physical constraints. We proceed to calculate $H^1(X,U)$ for $U = V, V^*, \wedge^2 V$ and $\wedge^2 V^*$ as well as $V \otimes V^*$ within the context of this example.

8.1 Calculation of $h^1(X,V)$ and $h^1(X,V^*)$

It follows from (118), (24) and (185) that the spectral data for V is given by

$$\mathcal{C}_V \in |5\sigma + \pi^*(12S + 15\mathcal{E})| \tag{186}$$

and

$$c_1(\mathcal{N}_V) = 5\sigma + \pi^*(3c_1(T\mathbb{F}_1)), \tag{187}$$

where, using (12),

$$c_1(T\mathbb{F}_1) = -c_1(K_{\mathbb{F}_1}) = 2S + 3\mathcal{E}. \tag{188}$$

It follows from (170) that

$$h^1(X,V) = h^0(\bar{c}_V, (\mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1})|_{\bar{c}_V}), \tag{189}$$

so we need to find expressions for both $\bar{c}_V$ and $(\mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1})|_{\bar{c}_V}$. The support curve $\bar{c}_V$ is easily calculated from (7), (10), (117), (186) and (188) to be

$$\bar{c}_V = 2S. \tag{190}$$

Furthermore, it follows from (187) and (188) that

$$c_1(\mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1}) = 5\sigma + \pi^*(4S + 6\mathcal{E}) \tag{191}$$

and, hence, using (7), (10) and (190) that

$$c_1(\mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1}) \cdot \bar{c}_V = -6. \tag{192}$$

Unfortunately, the support curve (190) is non-reduced, being scheme-theoretically twice the sphere S. The linear system $|2S|$ contains only one curve, this element is given by the unique global section of $\mathcal{O}_{\mathbb{F}_1}(2S)$, which vanishes along S to second order. Since such a curve is not smooth, we can not apply the above Riemann-Roch analysis. How then, can we proceed to evaluate $h^1(X, V)$? To do this, we first note that

$$S \simeq \mathbb{P}^1. \tag{193}$$

Now, for simplicity, denote the line bundle $\mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1}$ on X by

$$L = \mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1}. \tag{194}$$

When restricted to S, this bundle has degree

$$c_1(L)|_S = -3, \tag{195}$$

where we have used (7), (10) and (191). However, since S is a $\mathbb{P}^1$, it follows that $L|_S$ is none other than the line bundle

$$L|_S = \mathcal{O}_{\mathbb{P}^1}(-3). \tag{196}$$

Let us now invoke the following short exact sequence for the non-reduced scheme $2S$,

$$0 \to \mathcal{O}_S(-S) \to \mathcal{O}_{2S} \to \mathcal{O}_S \to 0 \ , \tag{197}$$

which, since S is a $\mathbb{P}^1$ in $\mathbb{F}_1$, becomes

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{2S} \to \mathcal{O}_{\mathbb{P}^1} \to 0 \ . \tag{198}$$

Tensoring this sequence with $L|_S$ in (196) gives us

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to L|_{2S} \to \mathcal{O}_{\mathbb{P}^1}(-3) \to 0 \ . \tag{199}$$

Now neither $\mathcal{O}_{\mathbb{P}^1}(-2)$ nor $\mathcal{O}_{\mathbb{P}^1}(-3)$ has global holomorphic sections, being of negative degree. This implies that $L|_{2S}$ also has no global sections. But

$$L|_{2S} = (\mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1})|_{\bar{c}_V}. \tag{200}$$

It then follows from (189) that

$$h^1(X, V) = h^0(\bar{c}_V, L|_{\bar{c}_V}) = 0 \ . \tag{201}$$

Thus by exploiting the exact sequence (197) on $S \simeq \mathbb{P}^1$, we have succeeded in computing $h^1(X, V)$.

As discussed previously, the dimensions of the first cohomology group of the dual bundle, $h^1(X, V^*)$, can be immediately computed from the index theorem result (106)

$$-h^1(X, V) + h^1(X, V^*) = 3. \tag{202}$$

It follows, using (201), that

$$h^1(X, V^*) = 3. \tag{203}$$

It is reassuring to calculate $h^1(X, V^*)$ directly using the method employed to compute $h^1(X, V)$. Hopefully, this will reproduce the result in (203). It follows from (122), (185) and (188) that the spectral data for V^* is given by

$$\mathcal{C}_{V^*} \in |5\sigma + \pi^*(12S + 15\mathcal{E})| \tag{204}$$

and

$$c_1(\mathcal{N}_{V^*}) = \pi^*(8S + 9\mathcal{E}). \tag{205}$$

Note that since $\mathcal{C}_{V^*}$ is in the same linear system as $\mathcal{C}_V$, it follows that

$$\bar{c}_{V^*} = \bar{c}_V = 2S. \tag{206}$$

Defining

$$L' = \mathcal{N}_{V^*} \otimes \pi^* K_{\mathbb{F}_1}, \tag{207}$$

we see, using (188) and (205), that

$$c_1(L') = \pi^*(6S + 6\mathcal{E}) \tag{208}$$

and, hence, using (10)

$$c_1(L')|_S = 0. \tag{209}$$

That is,

$$L'|_S = \mathcal{O}_{\mathbb{P}^1}. \tag{210}$$

Tensoring the short exact sequence in (198) with $L'|_S$ in (210) gives

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \to L'|_{2S} \to \mathcal{O}_{\mathbb{P}^1} \to 0\ . \tag{211}$$

Now, recall that

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0, \end{cases} \tag{212}$$

for integer n. It then follows from (211) and (212) that the number of global holomorphic sections of $L'|_{2S}$ is simply

$$h^0(S, \mathcal{O}_{\mathbb{P}^1}(1)) + h^0(S, \mathcal{O}_{\mathbb{P}^1}) = 3. \tag{213}$$

In deriving this result, we have used the fact that $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$. Noting that

$$L'|_{2S} = (\mathcal{N}_{V^*} \otimes \pi^* K_{\mathbb{F}_1})|_{\overline{c}_{V^*}} \ , \tag{214}$$

it follows from (189) that

$$h^1(X, V^*) = h^0(\overline{c}_{V^*}, (\mathcal{N}_{V^*} \otimes \pi^* K_{\mathbb{F}_1})|_{\overline{c}_{V^*}}) = 3 \ , \tag{215}$$

which is consistent with the index theorem result presented in (203).

8.2 Calculation of $h^1(X, \wedge^2 V)$ and $h^1(X, \wedge^2 V^*)$

We will calculate the term $h^1(X, \wedge^2 V^*)$ first because it will turn out to be computationally easier. We then use the index theorem (110) to compute $h^1(X, \wedge^2 V)$.

It follows from the first equation in (144) and (185) that the spectral cover for $\wedge^2 V^*$ is

$$\mathcal{C}_{\wedge^2 V^*} \in |10\sigma + \pi^* 3(12S + 15\mathcal{E})| \ . \tag{216}$$

Using (7), (117) and (188), we find that the associated support curve is given by

$$\overline{c}_{\wedge^2 V^*} \in [16S + 15\mathcal{E}] \ . \tag{217}$$

Note that every curve in this linear system is reducible, and decomposes into two generically non-intersecting components as follows.

$$\overline{c}_{\wedge^2 V^*} = C_1 \cup C_2, \tag{218}$$

where

$$C_1 \in [S], \quad C_2 \in [15(S + \mathcal{E})]. \tag{219}$$

Using (10) we see that

$$C_1 \cdot C_2 = 0. \tag{220}$$

So either C_1 and C_2 are disjoint, or C_2 must decompose further, with S a component of C_2. Since $\overline{c}_{\wedge^2 V^*}$ splits into C_1 and C_2, D also splits into two generically disjoint components,

$$D = D_1 \cup D_2, \tag{221}$$

where D_1 and D_2 are double covers of C_1 and C_2 respectively. It can be shown that

$$D_1 \in [2S], \tag{222}$$

and

$$D_2 \in [30(S + \mathcal{E}) + 90F] \ . \tag{223}$$

The restriction of $\pi_D : D \to C$ gives a double cover $\pi_{D_2} : D_2 \to C_2$ with branch divisor Br_2 and ramification R_2. However, D_1, as mentioned earlier, is actually contained in σ, and is, in fact, non-reduced so there is no divisor candidate for Br_1 and R_1.

The term we wish to compute is $h^1(X, \wedge^2 V^*)$. In principle, one could try to compute it in a manner similar to the calculation of $h^1(X, U)$ for $U = V, V^*$ above. However, this is not possible in the present case. Recall that in the previous section $\overline{c}_V = \overline{c}_{V^*} = 2S$, where $S \simeq \mathbb{P}^1$ is a rigid curve in $\mathbb{F}_1$. This allowed us to use a short exact sequence to relate the spectral line bundle twisted with $K_{\mathbb{F}_1}$ on $2S$ to line bundles on $\mathbb{P}^1$. The result then followed. Unfortunately, in the present case $\overline{c}_{\wedge^2 V}$ contains a component, $C_2 \in [15(S + \mathcal{E})]$, that is not a multiple copy of an isolated $\mathbb{P}^1$. This greatly complicates the problem, and makes a solution along the lines of the last section impossible. One might try to directly apply the Riemann-Roch theorem to the curve C_2. However, one finds that the associated line bundle lies outside the stable range. How then, can we proceed?

Let us first define the bundles

$$W = \mathcal{N}_{V^*}^{\otimes 2} \otimes \pi^* K_{\mathbb{F}_1} \tag{224}$$

and

$$Z = \mathcal{N}_{\wedge^2 V^*} \otimes \pi^* K_{\mathbb{F}_1} \ . \tag{225}$$

Then, the term we wish to compute is $h^0(\overline{c}_{\wedge^2 V^*}, Z|_{\overline{c}_{\wedge^2 V^*}})$. Applying (145) to $\pi_2 : D_2 \to C_2$ gives

$$\pi_{D_2 *}(\mathcal{N}_{V^*}^{\otimes 2}|_{D_2}) = \mathcal{N}_{\wedge^2 V^*}|_{C_2} \oplus \left(\mathcal{N}_{\wedge^2 V^*}|_{C_2} \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}) \right) \ . \tag{226}$$

Tensoring with $K_{\mathbb{F}_1}$ and taking h^0, this becomes

$$h^0(D_2, W|_{D_2}) = h^0(C_2, Z|_{C_2}) + h^0(C_2, (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2}) \ . \tag{227}$$

Putting this together with (170) and using the definitions (224) and (225), we obtain

$$\begin{aligned} h^1(X, \wedge^2 V^*) &= h^0(\overline{c}_{\wedge^2 V^*}, Z|_{\overline{c}_{\wedge^2 V^*}}) \\ &= h^0(C_1, Z|_{C_1}) + h^0(C_2, Z|_{C_2}) \end{aligned}$$

$$
\begin{aligned}
&= h^0(C_1, Z|_{C_1}) + h^0(D_2, W|_{D_2}) - h^0(C_2, (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2}) \\
&= -h^0(D_1, W|_{D_1}) + h^0(C_1, Z|_{C_1}) - h^0(C_2, (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2}) + h^0(D, W|_D)
\end{aligned} \tag{228}
$$

Therefore, we need to compute four terms, $h^0(D_1, W|_{D_1})$, $h^0(C_1, Z|_{C_1})$ $h^0(C_2, (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2})$ and $h^0(D, W|_D)$ in (228) to finish our calculation.

We begin by calculating the first term $h^0(D_1, W|_{D_1})$. Since $D_1 = 2S$, we see that this term can be computed exactly as in the previous section. First, we note that on S, W is simply

$$
W|_S = \mathcal{O}_{\mathbb{P}^1}(1). \tag{229}
$$

Next, recall from (198) that, for $S \simeq \mathbb{P}^1$, we have the exact sequence

$$
0 \to \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{2S} \to \mathcal{O}_{\mathbb{P}^1} \to 0 \ . \tag{230}
$$

Then, tensoring by $W|_S$ in (229) gives

$$
0 \to \mathcal{O}_{\mathbb{P}^1}(2) \to W|_{2S} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0. \tag{231}
$$

It now follows that

$$
h^0(D_1, W|_{D_1}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 5 \tag{232}
$$

where we have used (212).

Next, we compute the second of the four terms in (228), namely, $h^0(C_1, Z|_{C_1})$. Recalling that $C_1 = S$, this can also be readily computed. Note that

$$
Z|_{C_1} = (\mathcal{N}_{\wedge^2 V^*} \otimes \pi^* K_{\mathbb{F}_1})|_{C_1} \simeq W_S(-S) = \mathcal{O}_{\mathbb{P}^1}(2). \tag{233}
$$

Therefore, using (212), we find

$$
h^0(C_1, Z|_{C_1}) = 3. \tag{234}
$$

Now, we move on to the third requisite term $h^0(C_2, (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2})$ in (228). One way to do this is to try to use the Riemann-Roch theorem on the support curve as outlined in Subsection 7.5. In this regard, define the line bundle

$$
\mathcal{F} = (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2} \tag{235}
$$

and recall that

$$
\deg(\mathcal{F}) = c_1(\mathcal{F}) \ . \tag{236}
$$

The restriction of (144) to C_2 implies

$$c_1(\mathcal{N}_{\wedge^2 V^*}|_{C_2}) = c_1(\mathcal{N}_{V^*}) \cdot D_2 - \frac{R_2}{2} \tag{237}$$

where, by definition

$$c_1(\mathcal{O}_{C_2}(\frac{Br_2}{2})) = \frac{R_2}{2} \tag{238}$$

and R_2 are the ramification points on D_2. It follows that

$$\deg(\mathcal{F}) = c_1(\mathcal{N}_{V^*}^{\otimes 2}) \cdot D_2 + c_1(K_{\mathbb{F}_1}) \cdot C_2. \tag{239}$$

Evaluating (122) for the vacuum given in (185) and using (188) and the curves C_2 and D_2 presented in (219) and (223) respectively, we find

$$\deg(\mathcal{F}) = 225. \tag{240}$$

Now we compute the genus of C_2 using (153). Using (188) and (219), we find that

$$2g(C_2) - 2 = 180. \tag{241}$$

We note that the quantity

$$-\deg(\mathcal{F}) + 2g(C_2) - 2 = -45 < 0. \tag{242}$$

Thus, the line bundle $\mathcal{F}$ is in the stable range discussed in Subsection 7.5. Therefore, we can use (177) to determine $h^0(C_2, \mathcal{F})$. It follows from (177), (240) and (241) that

$$h^0(C_2, (Z \otimes \mathcal{O}_{C_2}(\frac{Br_2}{2}))|_{C_2}) = h^0(C_2, \mathcal{F}) = 225 - g(C_2) + 1 = 135. \tag{243}$$

This completes the calculation of the third of the four required terms.

Finally, we proceed to compute the remaining term $h^0(D, W|_D)$ in (228) to complete our computation. Unfortunately, evaluating this quantity is considerably more difficult. The term we need to compute is $h^0(D, W|_D)$. We will use the fact that $W|_D$ is a restriction of the global line bundle W on X. For such cases we have the technology to count global sections. In our particular example,

$$W = \pi^* \mathcal{O}_B(2\eta - 5c_1(T\mathbb{F}_1)) = \pi^* \mathcal{O}_B(14S + 15\mathcal{E}) \ , \tag{244}$$

where we have used (122) and (185). We proceed by considering the short exact sequence on $\mathcal{C}_V$

$$0 \to W|_{\mathcal{C}_V}(-D) \to W|_{\mathcal{C}_V} \to W|_D \to 0 \ , \tag{245}$$

where, using standard notation, we denote $W|_{\mathcal{C}_V} \otimes \mathcal{O}_{\mathcal{C}_V}(-D)$ by $W|_{\mathcal{C}_V}(-D)$. This induces a long exact sequence in cohomology

$$\begin{array}{cccccccc} 0 & \rightarrow & H^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \rightarrow & H^0(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow & \boxed{H^0(D, W|_D)} & \rightarrow \\ & \rightarrow & H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \xrightarrow{M_3} & H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow & H^1(D, W|_D) & \rightarrow \\ & \rightarrow & H^2(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \rightarrow & H^2(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow & H^2(D, W|_D) & \rightarrow \\ & \rightarrow & H^3(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \rightarrow & H^3(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow & H^3(D, W|_D) & \rightarrow \quad 0\ , \end{array} \tag{246}$$

where, in the third column, we have used the fact that for all $i \geq 0$,

$$H^i(\mathcal{C}_V, W|_D) \simeq H^i(D, W|_D) \tag{247}$$

because we are restricting W to D. Note that the cohomology group we are interested in, $H^0(D, W|_D)$, occurs in this sequence. For emphasis, we have boxed this term and indicated an explicit map, which we call M_3 and which will be essential in our calculation.

In general, for an exact sequence

$$\ldots \rightarrow A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \xrightarrow{d_3} A_4 \xrightarrow{d_4} A_5 \longrightarrow \ldots\ , \tag{248}$$

we have

$$\text{dim}A_3 = \text{dim}A_2 + \text{dim}A_4 - \text{rk}d_1 - \text{rk}d_4\ . \tag{249}$$

Therefore, (246) would give us

$$h^0(D, W|_D) = h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}) + h^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) - h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) - \text{rk}M_3\ . \tag{250}$$

We have used the fact that the rank of the map between $H^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))$ and $H^0(\mathcal{C}_V, W|_{\mathcal{C}_V})$ is simply equal to $h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))$ because this map, being the first in an exact sequence, is injective. We subsequently need to compute the cohomologies of $W|_{\mathcal{C}_V}(-D)$ and $W|_{\mathcal{C}_V}$ for which there are two more short exact sequences, both on X. These are

$$0 \rightarrow W(-\mathcal{C}_V) \rightarrow W \rightarrow W|_{\mathcal{C}_V} \rightarrow 0\ , \tag{251}$$

inducing the long exact sequence

$$\begin{array}{cccccccc} 0 & \rightarrow & H^0(X, W(-\mathcal{C}_V)) & \rightarrow & H^0(X, W) & \rightarrow & \boxed{H^0(\mathcal{C}_V, W|_{\mathcal{C}_V})} & \rightarrow \\ & \rightarrow & H^1(X, W(-\mathcal{C}_V)) & \xrightarrow{M_2} & H^1(X, W) & \rightarrow & H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow \\ & \rightarrow & H^2(X, W(-\mathcal{C}_V)) & \rightarrow & H^2(X, W) & \rightarrow & H^2(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow \\ & \rightarrow & H^3(X, W(-\mathcal{C}_V)) & \rightarrow & H^3(X, W) & \rightarrow & H^3(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \rightarrow \quad 0\ , \end{array} \tag{252}$$

as well as the sequence

$$0 \to W(-\mathcal{C}_V - D) \to W(-D) \to W|_{\mathcal{C}_V}(-D) \to 0 \ , \tag{253}$$

which induces the long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^0(X, W(-\mathcal{C}_V - D)) & \to & H^0(X, W(-D)) & \to & \boxed{H^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))} & \to & \\ & \to & H^1(X, W(-\mathcal{C}_V - D)) & \xrightarrow{M_1} & H^1(X, W(-D)) & \to & \boxed{H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))} & \to & \\ & \to & H^2(X, W(-\mathcal{C}_V - D)) & \to & H^2(X, W(-D)) & \to & H^2(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \to & \\ & \to & H^3(X, W(-\mathcal{C}_V - D)) & \to & H^3(X, W(-D)) & \to & H^3(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \to & 0 \ . \end{array} \tag{254}$$

Note that D, as defined in (126) and given in (128), is a curve in $\mathcal{C}_V$. However, it is not hard to show from (126) that D is the intersection of a divisor $\mathcal{C}_V - \sigma - \sigma_2$ on X with $\mathcal{C}_V$. Somewhat abusing notation, we will also denote this divisor by D. That is, let

$$D = \mathcal{C}_V - \sigma - \sigma_2 \ . \tag{255}$$

It is this divisor of X that occurs in the terms $W(-\mathcal{C}_V - D)$ and $W(-D)$ of (253) and (254), whereas D in $W|_{\mathcal{C}_V}(-D)$ is the curve (128). Which D we are referring to will be clear by context. As before, in the third column of (252) and (254), we have used the fact that for all $i \geq 0$,

$$H^i(X, W|_{\mathcal{C}_V}) \simeq H^i(\mathcal{C}_V, W|_{\mathcal{C}_V}) \tag{256}$$

and

$$H^i(X, W|_{\mathcal{C}_V}(-D)) \simeq H^i(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) \tag{257}$$

because of the restriction to $\mathcal{C}_V$. Again, we have boxed the requisite terms in (252) and (254) that we need in (250). We have also labeled two more maps, M_1 and M_2, which will be required in our calculation.

Now, the dimensions of the cohomology groups in the first two columns of the exact sequences (252) and (254) can be determined. We show how this is done in Appendix C. Using the techniques therein, we can fill in these dimensions as subscripts in the two sequences. We find that

$$\begin{array}{ccccccccc} 0 & \to & H^0(X, W(-\mathcal{C}_V))_0 & \to & H^0(X, W)_{135} & \to & \boxed{H^0(\mathcal{C}_V, W|_{\mathcal{C}_V})} & \to & \\ & \to & H^1(X, W(-\mathcal{C}_V))_{180} & \xrightarrow{M_2} & H^1(X, W)_{91} & \to & H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \to & \\ & \to & H^2(X, W(-\mathcal{C}_V))_2 & \to & H^2(X, W)_0 & \to & H^2(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \to & \\ & \to & H^3(X, W(-\mathcal{C}_V))_0 & \to & H^3(X, W)_0 & \to & H^3(\mathcal{C}_V, W|_{\mathcal{C}_V}) & \to & 0 \end{array} \tag{258}$$

and

$$\begin{array}{rlllll} 0 & \rightarrow & H^0(X, W(-\mathcal{C}_V - D))_0 & \rightarrow & H^0(X, W(-D))_0 & \rightarrow & \boxed{H^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))} & \rightarrow \\ & \rightarrow & H^1(X, W(-\mathcal{C}_V - D))_{84} & \xrightarrow{M_1} & H^1(X, W(-D))_{28} & \rightarrow & \boxed{H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))} & \rightarrow \\ & \rightarrow & H^2(X, W(-\mathcal{C}_V - D))_0 & \rightarrow & H^2(X, W(-D))_0 & \rightarrow & H^2(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \rightarrow \\ & \rightarrow & H^3(X, W(-\mathcal{C}_V - D))_{26} & \rightarrow & H^3(X, W(-D))_0 & \rightarrow & H^3(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) & \rightarrow \ 0\ . \end{array} \tag{259}$$

Applying (249) to (258), we have

$$h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}) = 135 + 180 - \mathrm{rk}\,(M_2) = 315 - \mathrm{rk}(M_2)\ . \tag{260}$$

Note that the matrix M_2 has dimensions

$$(M_2)_{91\times 180}\ . \tag{261}$$

Similarly, we can use (249) for (259) to obtain

$$h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) = 84 - \mathrm{rk}(M_1)\ , \tag{262}$$

and

$$h^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) = 28 - \mathrm{rk}(M_1)\ . \tag{263}$$

The matrix M_1 has the dimensions

$$(M_1)_{28\times 84}\ . \tag{264}$$

Substituting (260), (262) and (263) into (250) gives

$$h^0(D, W|_D) = 259 - \mathrm{rk}(M_2) - \mathrm{rk}(M_3)\ . \tag{265}$$

Let us study some limiting cases. It follows from (264) that M_1 has maximal rank 28 while its minimal rank is 0. Let us first assume that

$$\mathrm{rk}(M_1) = 28. \tag{266}$$

Then, it follows from (263) that

$$H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) = 0 \tag{267}$$

and, hence, from the sequence (246) that M_3 is the zero map. That is,

$$\mathrm{rk}(M_3) = 0\ . \tag{268}$$

Also, we see from (261) that the rank of M_2 is in the range

$$\mathrm{rk}(M_2) \in [0, 91] \ . \tag{269}$$

Throughout the remainder of this paper, we will use the symbol $[m, n]$ to indicate the range of integers from m to n. It does not imply that the quantity in question must assume all values in this range. Therefore expression (265) becomes a bound on the dimension of $h^0(D, W|_D)$ given by

$$h^0(D, W|_D) = 259 - \mathrm{rk}(M_2) \in [168, 259] \ . \tag{270}$$

On the other hand, let us assume that M_1 has its minimal rank. That is,

$$\mathrm{rk}(M_1) = 0. \tag{271}$$

In this case, it follows from (262) and (263) that

$$h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) = 84, \quad h^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D)) = 28. \tag{272}$$

Since $H^1(\mathcal{C}_V, W|_{\mathcal{C}_V}(-D))$ is not trivial, the mapping M_3 in sequence (246) is no longer the zero map. From (269), we know that $\mathrm{rk}(M_2) \in [0, 91]$. What are the possible ranks for M_3? First, assume M_2 has its maximal rank, that is,

$$\mathrm{rk}(M_2) = 91. \tag{273}$$

By inspecting (258), we see that, in this case,

$$h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}) = 224, \ h^1(\mathcal{C}_V, W|_{\mathcal{C}_V}) = 2. \tag{274}$$

It then follows from (246) that the rank of M_3 is in the range

$$\mathrm{rk}(M_3) \in [0, 2] \ . \tag{275}$$

On the other hand, if we assume

$$\mathrm{rk}(M_2) = 0 \ , \tag{276}$$

then (258) implies that

$$h^0(\mathcal{C}_V, W|_{\mathcal{C}_V}) = 315, \ h^1(\mathcal{C}_V, W|_{\mathcal{C}_V}) = 93 \ . \tag{277}$$

Then, from (246) and (263) we find

$$\mathrm{rk}(M_3) \in [0, 28]. \tag{278}$$

Putting this information together, we can conclude the following. When M_1 has its minimal rank, that is $\text{rk}(M_1) = 0$, then

$$\text{rk}(M_2) = 91 \Rightarrow \text{rk}(M_3) \in [0, 2], \quad h^0(D, W|_D) = 259 - \text{rk}(M_2) - \text{rk}(M_3) \in [166, 168] \tag{279}$$

and

$$\text{rk}(M_2) = 0 \Rightarrow \text{rk}(M_3) \in [0, 28], \quad h^0(D, W|_D) = 259 - \text{rk}(M_2) - \text{rk}(M_3) \in [231, 259]. \tag{280}$$

Having discussed these limiting ranges, we now need to explicitly compute the ranks of the maps $M_{i=1,2,3}$ to finish our calculation. This is a rather technical exercise and we leave the exposition of the method to Appendix D. We see from our discussion in the Appendix that the maps M_i depend on various complex parameters. These are the moduli associated with the vector bundle V. As we will show in the next section, there are 223 such moduli and, hence, the calculation of the ranks is rather complicated.

It will be very helpful if one can choose M_1 to have its maximal rank of 28. Then, by the above discussion, M_3 is the zero map and the requisite calculations are greatly simplified. To show that M_1 can, in fact, have maximal rank, at least for some subclass of vector bundle moduli, we explicitly compute the matrix M_1 given in (388), setting all moduli appearing in it to zero except those in the $m_{(3)0}$, $m_{(2)0}$ and $m_{(1)0}$ sub-blocks. The sub-blocks $m_{(R)i}$ of M_1 are defined in (395). If, in addition, we identify the moduli in these sub-blocks with a single modulus ϕ, the matrix M_1 can be explicitly written as

$$M_1 = \phi \left(\begin{bmatrix} \mathbb{1}_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{1}_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{3,3} & 0 & \mathbb{1}_{3,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{4,4} & 0 & \mathbb{1}_{4,4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{6,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{7,7} \end{bmatrix}_{28\times 35} \quad \begin{bmatrix} 0 \end{bmatrix}_{28\times 49} \right), \tag{281}$$

where $\mathbb{1}_{n,n}$ is the identity matrix of size n. For $\phi \neq 0$, the rank of this matrix is clearly 28, as required. Now turn on all other moduli in M_1. In general, it is not hard to show that, for generic values of these moduli, the rank of M_1 remains 28, decreasing only on specific loci of co-dimension one, or greater, in the moduli space. That is,

$$\text{rk}(M_1) = 28 \quad \text{generically.} \tag{282}$$

Therefore, not only can we choose M_1 to have rank 28 but it is the generic value. Then, as discussed above, the matrix M_3 is the zero matrix and we have

$$h^0(D, W|_D) = 259 - \text{rk}(M_2) \ . \tag{283}$$

In Appendix D, we explicitly construct M_2 and find that it is a 91×180 matrix depending on 139 complex moduli

$$\left\{\phi_p^{[(4)i]},\ \phi_q^{[(6)i]}\right\}, \qquad i = 1, \ldots, 9,\ p = 1, \ldots, i+1, \quad j = 3, \ldots, 12,\ q = 1, \ldots, j+1 \ . \tag{284}$$

The quantity $\text{rk}(M_2)$ is discussed in Appendix E. It is found to be extremely sensitive to the choice of these parameters. Figure 2, the table in (410) and Figure 3 show us that, depending on the values one chooses for these moduli,

$$\text{rk}(M_2) \in [28, 85] \tag{285}$$

and is, in fact, expected to attain all integer values between these bounds. The generic value is 85. It follows from this and (283) that

$$h^0(D, W|_D) = [174, 231] \ , \tag{286}$$

where 174 is the generic value.

We have now computed the last term $h^0(D, W|_D)$ of the four requisite terms in (228). Substituting (232), (234), (243) and (283) into (228), we obtain

$$h^1(X, \wedge^2 V^*) = 122 - \text{rk}(M_2) \ . \tag{287}$$

Therefore, substituting (285) into (287), we find at last that

$$h^1(X, \wedge^2 V^*) = [37, 94] \ , \tag{288}$$

where

$$h^1(X, \wedge^2 V^*) = 37 \text{ generically} \ . \tag{289}$$

Let us comment on what the result (288) means. One can make a plethora of choices when computing the ranks of the explicit maps M_1, M_2 and M_3. These correspond to the choice of the moduli on which each of these matrices depend. The final answer for $h^1(X, \wedge^2 V^*)$ will also depend, rather dramatically, on the values of these parameters. The generic result of 37 in (289) occurs for $\text{rk}(M_1) = 28$ and $\text{rk}(M_2) = 85$, which are their respective generic ranks. This means that as we move in the moduli space of the vector bundle V, we generically expect $h^1(X, \wedge^2 V^*)$ to be 37. However, as we hit special loci of co-dimension one or higher, as will be shown in Figure 1, the value of $h^1(X, \wedge^2 V^*)$ can jump to higher integers lying in the range (288). We conclude that the particle spectrum of the low-energy effective theory in heterotic compactifications depends crucially on the choice of the moduli of the vector bundle and can change as the values of the moduli change.

We have gone to great lengths to determine $h^1(X, \wedge^2 V^*)$ and to elucidate the fact that its value depends on the vector bundle moduli. To obtain $h^1(X, \wedge^2 V)$, we simply use the index relation (110)

$$h^1(X, \wedge^2 V) = h^1(X, \wedge^2 V^*) - 3 \ . \tag{290}$$

Given a value for $h^1(X, \wedge^2 V^*)$, this relation uniquely fixes $h^1(X, \wedge^2 V)$. It then follows from (288) and (289) that

$$h^1(X, \wedge^2 V) = [34, 91] \ , \tag{291}$$

where

$$h^1(X, \wedge^2 V) = 34 \text{ generically} \ . \tag{292}$$

It is important to note that even though the values of $h^1(X, \wedge^2 V)$ and $h^1(X, \wedge^2 V^*)$ depend dramatically upon the choice of moduli, their difference is constrained by the index theorem to be 3.

8.3 Calculation of $h^1(X, V \otimes V^*)$

As discussed earlier, we will compute the term $h^1(X, V \otimes V^*)$ using a different technique. We know that $h^1(X, V \otimes V^*)$ are the number of moduli associated with the vector bundle V. In Section 4 of [19], it was shown that this is equal to

$$h^1(X, V \otimes V^*) = (h^0(X, \mathcal{O}_X(\mathcal{C}_V)) - 1) + h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}). \tag{293}$$

Furthermore, $h^0(X, \mathcal{O}_X(\mathcal{C}_V))$ was computed and $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$ was shown to vanish in the case when the spectral cover $\mathcal{C}_V$ is positive. The conditions for positivity of a spectral cover

$$\mathcal{C}_V \in |n\sigma + \pi^*(aS + b\mathcal{E})| \tag{294}$$

over a base surface $B = \mathbb{F}_r$ were shown in [19] to be

$$b > a\, r - n(r-2), \quad a > 2n. \tag{295}$$

Under these circumstances, it was found that

$$h^1(X, V \otimes V^*) = \frac{n}{3}(4n^2 - 1) + nab - (n^2 - 2)(a + b) + ar(\frac{n^2}{2} - 1) - \frac{n}{2}ra^2 - a. \tag{296}$$

Can we use this expression to compute $h^1(X, V\otimes V^*)$ in the explicit example (185) being considered? Recall that, in this case,

$$\mathcal{C}_V \in |5\sigma + \pi^*(12S + 15\mathcal{E})| \tag{297}$$

is the spectral cover over $B = \mathbb{F}_1$. Putting the data $r = 1$, $n = 5$ and $(a, b) = (12, 15)$ into (295), we see that the requisite inequalities are violated. That is, in our specific example $\mathcal{C}_V$ is not a positive divisor and, hence, we can not use (296) to compute $h^1(X, V \otimes V^*)$. Unfortunately, for the $SU(5)$ GUT theories classified in the paper, this will often be the case. We, therefore, must use a different technique to compute $h^0(X, \mathcal{O}_X(\mathcal{C}_V))$ and $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$.

8.3.1 Moduli for the Spectral Cover

First, let us compute the term $h^0(X, \mathcal{O}_X(\mathcal{C}_V))$. These are the moduli associated with the spectral cover $\mathcal{C}_V$. To do this, first use (341) and (342) in Appendix C to push $H^0(X, \mathcal{O}_X(5\sigma+\pi^*(12S + 15\mathcal{E})))$ onto the base $\mathbb{F}_1$. We obtain

$$\begin{aligned} H^0(X, \mathcal{O}_X(5\sigma + \pi^*(12S + 15\mathcal{E}))) &= H^0(\mathbb{F}_1, \pi_*\mathcal{O}_X(5\sigma) \otimes \mathcal{O}_{\mathbb{F}_1}(12S + 15\mathcal{E})) \\ &= H^0(\mathbb{F}_1, (\mathcal{O}_{\mathbb{F}_1} \oplus \bigoplus_{i=2}^{5} \mathcal{O}_{\mathbb{F}_1}(-ic_1(T\mathbb{F}_1))) \otimes \mathcal{O}_{\mathbb{F}_1}(12S + 15\mathcal{E})) \end{aligned} \tag{298}$$

Since $\mathbb{F}_1$ is itself a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$, one can use (347) and (348) to push this result down further to $\mathbb{P}^1$. The answer is

$$\begin{aligned} & H^0(X, \mathcal{O}_X(5\sigma + \pi^*(12S + 15\mathcal{E}))) \\ = \; & \bigoplus_{i=3}^{15} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \oplus \bigoplus_{i=1}^{9} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \oplus \bigoplus_{i=0}^{6} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \\ & \oplus \bigoplus_{i=-1}^{3} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \oplus \bigoplus_{i=-2}^{0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)). \end{aligned} \tag{299}$$

Using (212), expression (299) gives

$$h^0(X, \mathcal{O}_X(5\sigma + \pi^*(12S + 15\mathcal{E}))) = 223. \tag{300}$$

8.3.2 Moduli for the Spectral Line Bundle

Next, we compute the term $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$. These are the moduli associated with the spectral line bundle, that is, the continuous moduli of the Picard group $H^1(\mathcal{C}_V, \mathcal{O}^*_{\mathcal{C}_V})$ of line bundles on $\mathcal{C}_V$. To show this, consider the exact sequence

$$\to H^1(\mathcal{C}_V, \mathbb{Z}) \to H^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) \to H^1(\mathcal{C}_V, \mathcal{O}^*_{\mathcal{C}_V}) \to H^2(\mathcal{C}_V, \mathbb{Z}) \to \;. \tag{301}$$

Now note that $H^1(\mathcal{C}_V, \mathbb{Z})$ and $H^2(\mathcal{C}_V, \mathbb{Z})$ are rigid lattices. Therefore, the only continuous moduli of $H^1(\mathcal{C}_V, \mathcal{O}^*_{\mathcal{C}_V})$ come from $H^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$. Hence, we need only to compute $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$. To do this, we use the short exact sequence

$$0 \to \mathcal{O}_X(-\mathcal{C}_V) \to \mathcal{O}_X \to \mathcal{O}_{\mathcal{C}_V} \to 0 \; , \tag{302}$$

which implies the long exact sequence

$$\to H^1(X, \mathcal{O}_X) \to H^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) \to H^2(X, \mathcal{O}_X(-\mathcal{C}_V)) \to H^2(X, \mathcal{O}_X) \to \; . \tag{303}$$

Now,

$$H^1(X, \mathcal{O}_X) = H^{0,1}_{\overline{\partial}}(X, \mathbb{C}), \quad H^2(X, \mathcal{O}_X) = H^{0,2}_{\overline{\partial}}(X, \mathbb{C}) \tag{304}$$

for the Dolbeault cohomolgy groups $H^{0,1}_{\overline{\partial}}(X, \mathbb{C})$ and $H^{0,1}_{\overline{\partial}}(X, \mathbb{C})$, both of which vanish for a Calabi-Yau threefold X. Therefore, (303) implies that

$$H^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) \simeq H^2(X, \mathcal{O}_X(-\mathcal{C}_V)) \; . \tag{305}$$

We can simplify this expression further by using Serre duality (98), which dictates that

$$H^2(X, \mathcal{O}_X(-\mathcal{C}_V)) \simeq H^1(X, \mathcal{O}_X(\mathcal{C}_V)) \tag{306}$$

on a Calabi-Yau threefold X. In summary, (305) and (306) together imply that the number of moduli associated with the spectral line bundle is

$$h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) = h^1(X, \mathcal{O}_X(\mathcal{C}_V)) \; . \tag{307}$$

Recalling from (186) that $\mathcal{C}_V \in |5\sigma + \pi^*(12S + 15\mathcal{E})|$, we see that the computation of (307) can be carried out using the techniques presented in Appendix C, in complete analogy with the above calculation for the moduli associated with the spectral cover. We find, after pushing everything onto the base $\mathbb{P}^1$, that we have

$$h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) = 1. \tag{308}$$

Substituting (300) and (308) into (293), we finally obtain

$$h^1(X, V \otimes V^*) = 223. \tag{309}$$

That is, in our explicit example, there are 223 vector bundle moduli.

8.3.3 Checking Against the Case of Positive Spectral Cover

As a check on our method, let us derive an expression for $h^1(X, V\otimes V^*)$ in the case of $B=\mathbb{F}_1$ and $\eta = aS + b\mathcal{E}$ with $a, b \in \mathbb{Z}_{\geq 0}$ where n, a, b satisfy (295), that is, when $\mathcal{C}_V$ is a positive spectral cover. In the case of positive spectral cover, it was shown in [19] that $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$ vanishes. Therefore, we need only compute $h^0(X, \mathcal{O}_X(\mathcal{C}_V))$. Then,

$$h^1(X, V\otimes V^*) = h^0(X, \mathcal{O}_X(\mathcal{C}_V) - 1\ . \tag{310}$$

First, recalling that n is always positive, we have

$$\begin{aligned} h^0(X, \mathcal{O}_X(n\sigma + \pi^*\eta)) &= h^0(B, (\mathcal{O}_B \oplus \bigoplus_{i=2}^{n} \mathcal{O}_B(-ic_1(T\mathbb{F}_1))) \otimes \mathcal{O}_B(\eta)) \\ &= h^0(B, \mathcal{O}_B(aS+b\mathcal{E})) + \sum_{i=2}^{n} h^0\left(B, \mathcal{O}_B((a-2i)S + (b-3i)\mathcal{E})\right), \end{aligned} \tag{311}$$

where we have used the expression for $c_1(T\mathbb{F}_1)$ in (12). Now, (295) clearly requires that $a > 0$, so the first term in (311) becomes

$$h^0(B, \mathcal{O}_B(aS+b\mathcal{E})) = h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{i=1}^{a} \mathcal{O}_{\mathbb{P}^1}(-i)) \otimes \mathcal{O}_{\mathbb{P}^1}(b))\ , \tag{312}$$

where we have used (348) to push down onto the base $\mathbb{P}^1$ of $\mathbb{F}_1$. Similarly, the second term in (311) becomes

$$\sum_{i=2}^{n} (h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b-3i)) + \sum_{j=1}^{a-2i} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((b-3i)-j)))\ . \tag{313}$$

Note that the bound $a-2i$ in the second sum is always positive since the positivity conditions for $\mathcal{C}_V$ in (295) require that $a > 2n$. Furthermore, the degrees $(b-3i)-j$ in the second term of (313) are always positive for $j \leq a-2i$ because (295) requires that $b > a+n$. Substituting (312) and (313) into (311) and using (212), we have at last

$$\begin{aligned} h^0(X, \mathcal{O}_X(\mathcal{C}_V)) &= (b+1) + \sum_{i=1}^{a}(b-i+1) + \sum_{i=2}^{n}((b-3i+1) + \sum_{j=1}^{a-2i}(b-3i-j+1)) \\ &= a + 2\,b - \frac{n}{3} - \frac{a^2\,n}{2} + a\,b\,n - \frac{a\,n^2}{2} - b\,n^2 + \frac{4\,n^3}{3}, \end{aligned} \tag{314}$$

which implies, using (310), that

$$h^1(X, V\otimes V^*) = \frac{n}{3}(4n^2-1) + nab - (n^2-2)(a+b) + a(\frac{n^2}{2} - 1) - \frac{n}{2}a^2 - 1\ . \tag{315}$$

This result agrees completely with (296) and Eq.4.39 of [19] which were computed by other methods for positive spectral cover. By this we are thus much assured.

8.4 Summary of the Particle Content

It is useful to summarize here the results of our calculation. We have compactified heterotic M-theory on an elliptic Calabi-Yau threefold whose base surface is $\mathbb{F}_1$ and on which there is a stable holomorphic vector bundle V with a structure group $G = SU(5)$ and spectral data

$$\mathcal{C}_V \in |5\sigma + \pi^*(12S + 15\mathcal{E})| \tag{316}$$

and

$$c_1(\mathcal{N}_V) = 5\sigma + \pi^*(3c_1(T\mathbb{F}_1)) \ . \tag{317}$$

This compactification satisfies the three physical constraints discussed in Section 5, that is, it is anomaly free, has three families of quarks and leptons and admits a gauge connection satisfying the hermitian Yang-Mills equation. The low energy GUT group is $H = SU(5)$. To distinguish the structure group of V from the GUT group, we will denote them by $SU(5)_G$ and $SU(5)_H$ respectively. The 5, $\overline{5}$, 10, $\overline{10}$ and 24 representations of $SU(5)_G$ are associated with the bundles V, V^*, $\wedge^2 V$, $\wedge^2 V^*$ and $V \otimes V^*$ respectively. The dimensions of the relevant cohomologies of these bundles were computed in the previous two sections and found to be

$SU(5)_G$	cohomology	spectrum
5	$h^1(X, V)$	0
$\overline{5}$	$h^1(X, V^*)$	3
10	$h^1(X, \wedge^2 V)$	$[34, 91]$, generically 34
$\overline{10}$	$h^1(X, \wedge^2 V^*)$	$[37, 94]$, generically 37
24	$h^1(X, V \otimes V^*)$	223

(318)

The low energy theory is a four-dimensional $N = 1$ supersymmetric GUT theory with gauge group $SU(5)_H$. The expressions for the spectrum of chiral superfields transforming as the $\overline{10}$, 10, 5, $\overline{5}$, and 1 of $SU(5)_H$ were discussed in Section 7 and given in (96) and (97). Combining these expressions, we have the following. Trivially, there is one vector supermultiplet transforming as the adjoint 78 representation of $SU(5)_H$. That is,

$$n_{78} = 1 \ . \tag{319}$$

The number of chiral supermultiplets in the $\overline{10}$ and 10 representations of $SU(5)_H$ are

$$n_{\overline{10}} = 0, \quad n_{10} = 3 \tag{320}$$

respectively. The number of chiral supermultiplets in the $\overline{5}$ representation of $SU(5)_H$ depends on the values of the vector bundle moduli. Generically, we find that

$$n_{\overline{5}} = 37. \tag{321}$$

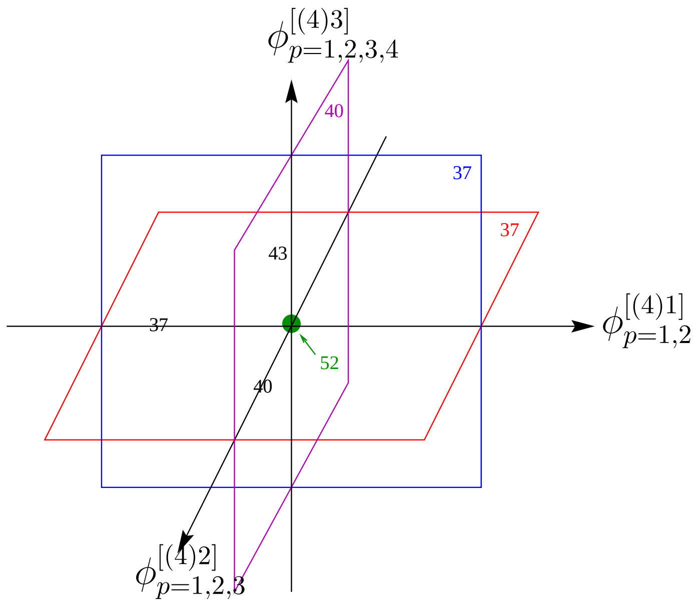

Figure 1: A subspace of the moduli space $\mathcal{M}$ of ϕ's spanned by $\phi^{[(4)1]}_{p=1,2}$, $\phi^{[(4)2]}_{q=1,2,3}$ and $\phi^{[(4)3]}_{r=1,2,3,4}$. Generically, in the bulk, $n_{\overline{5}} = 37$, its minimal value. As we restrict to various planes and intersections thereof, we are confining ourselves to special sub-spaces of co-dimension one or higher. In these subspaces, the value of $n_{\overline{5}}$ can increase dramatically.

However, on loci of co-dimension one or higher in the moduli space this value can abruptly jump, spanning the range

$$n_{\overline{5}} \in [37, 94] \ . \tag{322}$$

We expect that each integer value in this range is realized on some subset of moduli space. As a graphic example of this phenomenon, we show in Figure 1 a nine-dimensional region of vector bundle moduli space discussed in Appendix E. Note that for a generic point in this space, $n_{\overline{5}} = 37$. However, on various sub-planes of co-dimension one or higher $n_{\overline{5}}$ jumps, taking the values $n_{\overline{5}} = 37, 40, 43$ and 52. These numbers are obtained using (287) and the results in table (410).

The index theorem tells us that the number of chiral supermultiplets transforming in the 5 representation of $SU(5)_H$ is given by

$$n_5 = n_{\overline{5}} - 3. \tag{323}$$

Therefore, generically

$$n_5 = 34. \tag{324}$$

However, it follow from (322) and (323) that this number can jump, spanning the range

$$n_5 \in [34, 91] \ . \tag{325}$$

Note, however, that the index theorem guarantees that at every point in moduli space

$$n_{\bar{5}} - n_5 = 3 \ . \tag{326}$$

Finally, the number of chiral superfields transforming as singlets under $SU(5)_H$, that is, the number of vector bundle moduli, is given by

$$n_1 = 223 \ . \tag{327}$$

We have succeeded, therefore, in computing the exact particle spectrum of our $SU(5)$ GUT theory. Rather remarkably, we find that, although the difference $n_{\bar{5}} - n_5$ is fixed by the three family condition to be 3, the individual values of n_5 and $n_{\bar{5}}$ depend on the location in vector bundle moduli space at which they are evaluated.

9 Conclusions

We have shown, for general heterotic vacua, that the calculation of the particle spectrum consists of computing the sheaf cohomology of five vector bundles: $V, V^*, \wedge^2 V, \wedge^2 V^*, V \otimes V^*$. Among these, the cohomology group $H^1(X, V \otimes V^*)$ has a topological interpretation as the number of deformation moduli of the vector bundle V and, therefore, is always deformation invariant. Hence, it cannot jump when the bundle moduli are varied continuously. However, no such topological interpretation is available for the cohomologies of the remaining bundles. In fact, we have shown that, as V varies continuously, the cohomologies of $\wedge^2 V$ and $\wedge^2 V^*$ do jump in the particular theory under consideration in this paper.

The novelty of these results is seen when contrasted with the standard embedding. In this latter case, there are two distinct ways to deform the vector bundle. The first is to deform the Calabi-Yau threefold X while keeping $V = TX$. In this case, jumps in the spectrum can never occur since all the cohomologies in question do have a topological interpretation. By definition, in the standard embedding we have $V = \wedge^2 V^* = TX = \Omega^2_X$ and $V^* = \wedge^2 V = T^*X = \Omega^1_X$. Hence, the cohomologies are all of the form $H^i(\Omega^j_X)$, $j = 1, 2$ and $i = 0, \ldots, 3$ where Ω^j_X is the sheaf of holomorphic j-forms on X. On our Calabi-Yau threefold X these are topological invariants since their Hodge numbers can be related to the Betti numbers as

$$h^{1,1} = b^2, \qquad 2h^{2,1} + 2 = b^3. \tag{328}$$

This explains why jumps in the spectrum as the moduli of the Calabi-Yau threefold are varied were never observed in heterotic compactifications based on the standard embedding. The second way to deform the vector bundle is to start with the standard embedding $V = TX$ on a fixed Calabi-Yau threefold and to deform its tangent bundle, as in [33]. In this case, it is not excluded, and is in fact quite likely, that jumps in the particle spectrum will occur.

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Appendices

A Chern and Todd Classes

For convenience, we remind the reader of the expansion of the Chern and Todd classes for a vector bundle U on a complex manifold X.

$$\begin{aligned} \mathrm{td}(U) &= 1 + \mathrm{td}_1(U) + \mathrm{td}_2(U) + \mathrm{td}_3(U) + \dots \\ \mathrm{ch}(U) &= \mathrm{ch}_0(U) + \mathrm{ch}_1(U) + \mathrm{ch}_2(U) + \mathrm{ch}_3(U) + \dots \end{aligned} \tag{329}$$

with

$$\mathrm{td}_1(U) = \frac{1}{2}c_1(U), \quad \mathrm{td}_2(U) = \frac{1}{12}(c_2(U) + c_1(U)^2), \quad \mathrm{td}_3(U) = \frac{1}{24}(c_1(U)c_2(U)) \ , \dots \tag{330}$$

and

$$\begin{aligned} \mathrm{ch}_0(U) &= \mathrm{rk}(U), \quad \mathrm{ch}_1(U) = c_1(U), \quad \mathrm{ch}_2(U) = \frac{1}{2}(c_1(U)^2 - 2c_2(U)), \\ \mathrm{ch}_3(U) &= \frac{1}{6}(c_1(U)^3 - 3c_1(U)c_2(U) + 3c_3(U)) \ \dots . \end{aligned} \tag{331}$$

We will make frequent use of these formulas.

B Chern Classes of Antisymmetric Products

In this Appendix, we obtain the expressions for the Chern classes of the antisymmetric product $\wedge^2 V$ of a rank n vector bundle V on a threefold X. Using the splitting principle, let us first decompose V into n line bundles L_i as

$$V = \bigoplus_{i=1}^{n} L_i \ . \tag{332}$$

Then, we have that

$$\mathrm{ch}(V) = \sum_{i=1}^{n} e^{x_i} \tag{333}$$

where

$$x_i = c_1(L_i) \ . \tag{334}$$

Since X is a threefold, (333) can be expanded as

$$\mathrm{ch}(V) \ = \ \sum_{i=1}^{n} 1 + x_i + \frac{1}{2}x_i^2 + \frac{1}{6}x_i^3 \ . \tag{335}$$

Therefore, we can read off from (335) that

$$\mathrm{ch}_0(V) = \mathrm{rk}(V) = n, \quad \mathrm{ch}_1(V) = \sum_{i=1}^{n} x_i, \quad \mathrm{ch}_2(V) = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \quad \mathrm{ch}_3(V) = \frac{1}{6}\sum_{i=1}^{n} x_i^3 \ . \tag{336}$$

The result we are after is

$$\mathrm{ch}(\wedge^2 V) = \sum_{i<j}^{n} e^{x_i} e^{x_j} \ , \tag{337}$$

which, using (335), becomes

$$\begin{aligned}
\mathrm{ch}(\wedge^2 V) &= \sum_{i<j}^{n} 1 + (x_i + x_j) + \frac{1}{2}(x_i + x_j)^2 + \frac{1}{6}(x_i + x_j)^3 \\
&= \frac{n(n-1)}{2} + (n-1)\sum_{i=1}^{n} x_i + \frac{1}{4}\left(2n\sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j - 4\sum_{i=1}^{n} x_i^2\right) + \\
&\qquad \frac{1}{12}\left(2n\sum_{i=1}^{n} x_i^3 + 6\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} x_j - 8\sum_{i=1}^{n} x_i^3\right) \\
&= \left[\frac{n(n-1)}{2}\right] + \Big[(n-1)\,\mathrm{ch}_1(V)\Big] + \left[(n-2)\mathrm{ch}_2(V) + \frac{1}{2}\mathrm{ch}_1(V)^2\right] + \\
&\qquad \Big[(n-4)\mathrm{ch}_3(V) + \mathrm{ch}_1(V)\mathrm{ch}_2(V)\Big] \ .
\end{aligned} \tag{338}$$

Using the relations given in (329) and (331), we conclude that the Chern classes of $\wedge^2 V$ are

$$\begin{aligned}
\mathrm{rk}(\wedge^2 V) &= \frac{n(n-1)}{2}, \\
c_1(\wedge^2 V) &= (n-1)c_1(V), \\
c_2(\wedge^2 V) &= \frac{(n-1)(n-2)}{2} c_1(V)^2 + (n-2)c_2(V), \\
c_3(\wedge^2 V) &= \frac{(n-1)(n-2)(n-3)}{6} c_1(V)^3 + (n-2)^2 c_1(V)c_2(V) + (n-4)c_3(V).
\end{aligned} \tag{339}$$

C Determining $H^i(X, \mathcal{O}_X(n\sigma) \otimes \pi^* L)$

In this Appendix, we determine, using the Leray spectral sequence, the cohomology groups $H^i(X, T)$ for line bundles of the form $T = \mathcal{O}_X(n\sigma) \otimes \pi^* L$ where L is some line bundle on the base B. In particular, we will be interested in the dimensions of these groups and the explicit maps between them. We will use the Leray spectral sequence to reduce the calculation of the cohomology on X to that on the base B. Then, we specialize to the case when $B = \mathbb{F}_1$, which is itself a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$. In this case, we use the Leray spectral sequence again to reduce the cohomology on B to that on $\mathbb{P}^1$ with which we are familiar. In all, for $B = \mathbb{F}_1$, we can compute the cohomology groups $H^i(X, \mathcal{O}_X(n\sigma) \otimes \pi^* L)$ in general by reducing them to direct sums of cohomologies over $\mathbb{P}^1$.

For $\pi : X \to B$, with B being a surface and T a line bundle, the Leray spectral sequence becomes the long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^1(B, \pi_* T) & \to & H^1(X, T) & \to & H^0(B, R^1\pi_* T) & \to & \\ & \to & H^2(B, \pi_* T) & \to & H^2(X, T) & \to & H^1(B, R^1\pi_* T) & \to & \\ & \to & H^3(B, \pi_* T) & \to & H^3(X, T) & \to & H^2(B, R^1\pi_* T) & \to & 0. \end{array} \tag{340}$$

For T of the form $T = \mathcal{O}_X(n\sigma) \otimes \pi^* L$, (340) gives us

$n > 0$	$R^1\pi_* T = 0$	$\begin{cases} H^i(X, T) = H^i(B, \pi_* \mathcal{O}_X(n\sigma) \otimes L), \ i = 0, 1, 2 \\ H^3(X, T) = 0 \end{cases}$
$n < 0$	$\pi_* T = 0$	$\begin{cases} H^0(X, T) = 0 \\ H^i(X, T) = H^{i-1}(B, R^1\pi_* T) \\ \qquad = H^{i-1}(B, R^1\pi_*(\mathcal{O}_X(n\sigma)) \otimes L), \ i = 1, 2, 3 \end{cases}$
$n = 0$	$\pi_* T = L$, $R^1\pi_* T = K_B \otimes L$	$\begin{cases} H^0(X, \pi^* L) = H^0(B, L) \\ H^1(X, \pi^* L) = H^1(B, L) \oplus H^0(B, K_B \otimes L) \ \text{ if } \ H^2(B, L) = 0 \end{cases}$

(341)

where, for $n > 0$ and $n < 0$, we have used the fact that

$$\begin{aligned} \pi_*(\mathcal{O}_X(n\sigma)) &= \begin{cases} \mathcal{O}_B \oplus \mathcal{O}_B(-2c_1(TB)) \oplus \ldots \oplus \mathcal{O}_B(-nc_1(TB)) & \text{for} \quad n > 0 \\ 0 & \text{for} \quad n < 0 \end{cases} \\ R^1\pi_*(\mathcal{O}_X(n\sigma)) &= \begin{cases} 0 & \text{for} \quad n > 0 \\ \mathcal{O}_B((-n-1)c_1(TB)) \oplus \ldots \oplus \mathcal{O}_B(c_1(TB)) \oplus \mathcal{O}_B(-c_1(TB)) & \text{for} \quad n < 0 \ . \end{cases} \end{aligned} \tag{342}$$

In the last case of $n = 0$ in (341), we have used the relation

$$R^1\pi_* \mathcal{O}_X = K_B \tag{343}$$

and the fact that (340) reduces, for $n = 0$, to

$$0 \to H^1(B, L) \to H^1(X, \pi^* L) \to H^0(B, R^1\pi_* \mathcal{O}_X \otimes L) \to H^2(B, L) \to \ldots \ . \tag{344}$$

Now the Hirzebruch surface $B = \mathbb{F}_1$ is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$. Therefore, there is a projection map

$$\beta : B \to \mathbb{P}^1 \ . \tag{345}$$

Any line bundle L on $B = \mathbb{F}_1$ can be written, by (11), as

$$L = \mathcal{O}_B(aS + b\mathcal{E}) \ , \quad a, b \in \mathbb{Z} \ . \tag{346}$$

A similar application as (340), replacing X, B and T by B, $\mathbb{P}^1$ and $\mathcal{O}_B(aS+b\mathcal{E})$ respectively, gives us

$$\begin{aligned} H^0(B, \mathcal{O}_B(aS + b\mathcal{E})) &= H^0(\mathbb{P}^1, \beta_*\mathcal{O}_B(aS) \otimes \mathcal{O}_{\mathbb{P}^1}(b)), \\ H^1(B, \mathcal{O}_B(aS + b\mathcal{E})) &= H^0(\mathbb{P}^1, R^1\beta_*\mathcal{O}_B(aS) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) \oplus H^1(\mathbb{P}^1, \beta_*\mathcal{O}_B(aS) \otimes \mathcal{O}_{\mathbb{P}^1}(b)), \\ H^2(B, \mathcal{O}_B(aS + b\mathcal{E})) &= H^1(\mathbb{P}^1, R^1\beta_*\mathcal{O}_B(aS) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) \ . \end{aligned} \tag{347}$$

where we have, for $\beta_*\mathcal{O}(aS)$ and $R^1\beta_*\mathcal{O}(aS)$,

	$\beta_*\mathcal{O}(aS)$	$R^1\beta_*\mathcal{O}(aS)$
$a \geq 0$	$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$	0
$a < 0$	0	$\mathcal{O}_{\mathbb{P}^1}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)$

(348)

Combining the results in (341) and (347) gives us a method of expressing $H^i(X, \mathcal{O}_X(n\sigma) \otimes \pi^*L)$ in terms of a much more familiar object which can be handled with ease, namely $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$.

Let us demonstrate this technique by computing $H^1(X, W)$ for a specific example. Choose $n = 0$ and

$$L = \mathcal{O}_B(14S + 15\mathcal{E}). \tag{349}$$

Note this corresponds to choosing

$$T = W \ , \tag{350}$$

where the line bundle W is defined in (244). That is, the sheaf cohomology group we wish to consider is $H^1(X, \pi^*L)$. First, we note that

$$H^2(B, L) = H^1(\mathbb{P}^1, R^1\beta_*\mathcal{O}_B(14S) \otimes \mathcal{O}_{\mathbb{P}^1}(15)) = 0, \tag{351}$$

since $R^1\beta_*\mathcal{O}_B(14S)$ vanishes by (348). Therefore, using (341), we conclude that

$$H^1(X, \pi^*\mathcal{O}_B(14S+15\mathcal{E})) = H^1(B, \mathcal{O}_B(14S+15\mathcal{E})) \oplus H^0(B, \mathcal{O}_B(-c_1(\mathbb{F}_1)+14S+15\mathcal{E})). \tag{352}$$

Using (348), this becomes

$$H^0(\mathbb{P}^1, R^1\beta_*\mathcal{O}_B(14S) \otimes \mathcal{O}_{\mathbb{P}^1}(15)) \oplus H^1(\mathbb{P}^1, \beta_*\mathcal{O}_B(14S) \otimes \mathcal{O}_{\mathbb{P}^1}(15)) \oplus H^0(B, \beta_*\mathcal{O}_B(12S) \otimes \mathcal{O}_{\mathbb{P}^1}(12)). \tag{353}$$

Upon simplifying (353), we at last have

$$H^1(X,W) \simeq \bigoplus_{i=0}^{12} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)). \tag{354}$$

We can similarly determine the other cohomology groups for W, in summary, one finds that

$$\begin{aligned} H^0(X,W) &\simeq \bigoplus_{i=0}^{15} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \ , \\ H^1(X,W) &\simeq \bigoplus_{i=0}^{12} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \ , \\ H^2(X,W) &\simeq H^3(X,W) = 0 \ . \end{aligned} \tag{355}$$

Using the dimensions of the cohomology groups on $\mathbb{P}^1$ given in (212), we readily find that

$$h^0(X,W) = 135,\ h^1(X,W) = 91,\ h^2(X,W) = h^3(X,W) = 0 \ . \tag{356}$$

D Constructing the Maps M_i Explicitly

In order to construct the linear maps M_i in (246), (252) and (254), we first need to determine the relevant cohomology groups $H^i(X,T)$ as vector spaces. This was done in Appendix C. Next, we will construct explicit bases for these vector spaces. Finally, we write the matrix representative for M_i by finding the multiplication rules which transform a basis of the domain to a basis for the range.

We illustrate our technique with M_1 which, we recall from (259), is the following mapping

$$H^1(X, W(-\mathcal{C}_V - D))_{84} \xrightarrow{M_1} H^1(X, W(-D))_{28} \ . \tag{357}$$

The method we use is similar to that of [32]. We will need the bases for three vector spaces, the domain $H^1(X, W(-\mathcal{C}_V - D))_{84}$, the range $H^1(X, W(-D))_{28}$ and $H^0(X, \mathcal{O}_X(\mathcal{C}_V))$. This last space classifies the mapping M_1. To see this, note that our map on cohomology

$$M_1 \in \mathrm{Hom}(H^1(X, W(-\mathcal{C}_V - D)), H^1(X, W(-D))) \tag{358}$$

is induced by a sheaf map

$$\tilde{M}_1 : W(-\mathcal{C}_V - D) \to W(-D) \tag{359}$$

and both are given by the cup product with a class in

$$H^0(X, W(-\mathcal{C}_V - D)^* \otimes W(-D)) = H^0(X, \mathcal{O}_X(\mathcal{C}_V)) \; . \tag{360}$$

We will illustrate our technique with the space $H^1(X, W(-D))_{28}$, the range of the map M_1. From (244), we have

$$W(-D) = \mathcal{O}_X(-\sigma + \pi^*(8S + 9\mathcal{E})). \tag{361}$$

The method of expressing a cohomology group of this form in terms of those on the base $\mathbb{F}_1$ and then on the base $\mathbb{P}^1$ of $\mathbb{F}_1$ has already been presented in Appendix C using the Leray spectral sequence. By (341), (342), (347) and (12), we have

$$\begin{array}{c} H^1(X, W(-D)) = H^1(X, \mathcal{O}_X(-\sigma + \pi^*(8S + 9\mathcal{E}))) \\ \Big\downarrow \pi_* \\ H^0(B, \mathcal{O}_B(6S + 6\mathcal{E})) \\ \Big\downarrow \beta_* \\ H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}) \; . \end{array} \tag{362}$$

We now describe the vector space on $\mathbb{P}^1$ explicitly by defining

$$B_{(3)k} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \tag{363}$$

for $k = 0, 1, \ldots, 6$ where

$$\dim B_{(3)k} = k + 1. \tag{364}$$

The vector space at the bottom of (362), the cohomology group on $\mathbb{P}^1$, can now be written as

$$\bigoplus_{k=1}^{6} B_{(3)k}. \tag{365}$$

Next, we pull back this space to $\mathbb{F}_1$ using β^* and define

$$b_{(3)k} = \beta^* B_{(3)k} \; . \tag{366}$$

Note that $b_{(3)k}$ is the space of sections of $\mathcal{O}_B(k\mathcal{E})$. That is,

$$b_{(3)k} = H^0(B, \mathcal{O}_B(k\mathcal{E})). \tag{367}$$

In order to embed this as a subspace of $H^0(B, \mathcal{O}_B(6S+6\mathcal{E}))$, each element of $H^0(B, \mathcal{O}_B(k\mathcal{E}))$ must be multiplied by a fixed section of $\mathcal{O}_B(6S + 6\mathcal{E} - k\mathcal{E})$. Defining

$$s_{(3)k} = 3c_1(T\mathbb{F}_1) - k\mathcal{E} \ , \tag{368}$$

the reader can readily verify, using (12), that

$$s_{(3)k} - 3\mathcal{E} = 6S + 6\mathcal{E} - k\mathcal{E} \ . \tag{369}$$

Let us denote the fixed section by

$$\tilde{s}_{(3)k} \in H^0(B, \mathcal{O}_B(s_{(3)k} - 3\mathcal{E})) \tag{370}$$

and the space of sections of $\mathcal{O}_B(k\mathcal{E})$ multiplied by $\tilde{s}_{(3)k}$ as $b_{(3)k}\tilde{s}_{(3)k}$. Then, we can write the middle term in (362) as

$$H^0(B, \mathcal{O}_B(6S + 6\mathcal{E})) = \bigoplus_{k=0}^{6} b_{(3)k}\tilde{s}_{(3)k} \ . \tag{371}$$

Finally, we pull back this space to X using π^*. We find that

$$H^1(X, W(-D)) = \bigoplus_{k=0}^{6} \hat{b}_{(3)k}\hat{\tilde{s}}_{(3)k}\tilde{a}, \tag{372}$$

where

$$a = -\sigma + \pi^*(c_1(\mathbb{F}_1)), \quad \hat{b}_{(3)k} = \pi^*(b_{(3)k}), \quad \hat{\tilde{s}}_{(3)k} = \pi^*(\tilde{s}_{(3)k}) \tag{373}$$

and

$$\tilde{a} \in H^0(X, \mathcal{O}_X(a)) \ . \tag{374}$$

The factor $\tilde{a}$ arises for the same reason as the $\tilde{s}_{(3)k}$ factor and we use similar notation. That is, $\tilde{a}$ is required so that each term on the right hand side of (372) is a subspace of $H^1(X, W(-D))$. Specifically, the notation in (372) indicates that one should take the element $\tilde{a}$ of $H^1(X, \mathcal{O}_X(a))$ and multiply each element of $\bigoplus_{k=0}^{6} \hat{b}_{(3)k}\hat{\tilde{s}}_{(3)k}$ by it. In summary, for the term $H^1(X, W(-D))$, we have

$$
\begin{array}{c}
H^1(X, W(-D)) = \bigoplus_{k=0}^{6} \hat{b}_{(3)k} \hat{\tilde{s}}_{(3)k} \tilde{a} \\
\pi^* \Big\uparrow \\
H^0(B, \mathcal{O}_B(6S + 6\mathcal{E})) = \bigoplus_{k=0}^{6} b_{(3)k} \tilde{s}_{(3)k} \\
\beta^* \Big\uparrow \\
\bigoplus_{k=0}^{6} B_{(3)k}.
\end{array}
\tag{375}
$$

The above procedure can be repeated for the other two terms, the domain $H^1(X, W(-\mathcal{C}_V - D))$ and the mapping $M_1 \in H^0(X, \mathcal{O}_X(\mathcal{C}_V))$. For $H^1(X, W(-\mathcal{C}_V - D))$, we have the following decomposition.

$$
\begin{array}{ll}
H^1(X, W(-\mathcal{C}_V - D)) = \bigoplus_{Q=0}^{3} \bigoplus_{j=Q}^{3Q} \hat{c}_{(Q)j} \hat{\tilde{s}}_{(Q)j} \tilde{a}_{(Q)}, & \begin{array}{l} \hat{c}_{(Q)j} = \pi^* c_{(Q)j}, \\ \hat{\tilde{s}}_{(Q)j} = \pi^* \tilde{s}_{(Q)j}, \\ a_{(Q)} = -6\sigma - \pi^*((Q+2)c_1(T\mathbb{F}_1)), \\ \tilde{a}_{(Q)} \in H^0(X, \mathcal{O}_X(a(Q))) \end{array} \\
\pi^* \Big\uparrow & \\
\bigoplus_{Q=0}^{3} \bigoplus_{j=Q}^{3Q} c_{(Q)j} \tilde{s}_{(Q)j}, & \begin{array}{l} c_{(Q)j} = \beta^* C_{(Q)j}, \\ s_{(Q)j} = Qc_1(T\mathbb{F}_1) - j\mathcal{E}, \\ \tilde{s}_{(Q)j} \in H^0(B, \mathcal{O}_B(s_{(Q)j})) \end{array} \\
\beta^* \Big\uparrow & \\
\bigoplus_{Q=0}^{3} \bigoplus_{j=Q}^{3Q} C_{(Q)j}, & C_{(Q)j} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j)).
\end{array}
\tag{376}
$$

Finally, the space $H^0(X, \mathcal{O}_X(\mathcal{C}_V))$, in which the map M_1 lives, has the basis

$$
\begin{array}{lcl}
H^0(X, \mathcal{O}_X(\mathcal{C}_V)) = \bigoplus_{R=1,2,3,4,6} \bigoplus_{i=R-3}^{3R-3} \hat{m}_{(R)i} \hat{\tilde{s}}_{(R)i} \tilde{A}_{(R)}, & & \begin{array}{l} \hat{m}_{(R)i} = \pi^* m_{(R)i}, \\ \hat{\tilde{s}}_{(R)i} = \pi^* s_{(R)i}, \\ A_{(R)i} = 5\sigma + \pi^*((6-R)c_1(T\mathbb{F}_1)), \\ \tilde{A}_{(R)i} \in H^0(X, \mathcal{O}_X(A_{(R)i})) \end{array} \\
\uparrow \pi^* & & \\
\bigoplus_{R=1,2,3,4,6} \bigoplus_{i=R-3}^{3R-3} m_{(R)i} \tilde{s}_{(R)i} \ , & & \begin{array}{l} m_{(R)i} = \beta^* M_{(R)i}, \\ s_{(R)i} = Rc_1(T\mathbb{F}_1) - i\mathcal{E}, \\ \tilde{s}_{(R)i} \in H^0(B, \mathcal{O}_B(s_{(R)i})) \end{array} \\
\uparrow \beta^* & & \\
\bigoplus_{R=1,2,3,4,6} \bigoplus_{i=R-3}^{3R-3} M_{(R)i}, & & M_{(R)i} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)).
\end{array}
\tag{377}
$$

We have now explicitly constructed the bases for the vector spaces of concern. In particular, $M_1 \in H^0(X, \mathcal{O}_X(\mathcal{C}_V))$ given in (377) must map $H^1(X, W(-\mathcal{C}_V - D))_{84}$ given in (376) linearly to $H^1(X, W(-D))_{28}$ given in (375). The matrix for M_1 is straight-forward to construct. We multiply the basis in (376) with the matrix elements in (377) and demand that the result be in (375). More specifically, we must guarantee that

$$\left(\hat{m}_{(R)i} \hat{\tilde{s}}_{(R)i} \tilde{A}_{(R)} \right) \cdot \left(\hat{c}_{(Q)j} \hat{\tilde{s}}_{(Q)j} \tilde{a}_{(Q)} \right) = \hat{b}_{(3)k} \hat{\tilde{s}}_{(3)k} \tilde{a} \ . \tag{378}$$

This requires that

$$\tilde{A}_{(R)} \cdot \tilde{a}_{(Q)} = \tilde{a} \ , \tag{379}$$

$$\hat{\tilde{s}}_{(R)i} \cdot \hat{\tilde{s}}_{(Q)j} = \hat{\tilde{s}}_{(3)k} \tag{380}$$

and

$$\hat{m}_{(R)i} \cdot \hat{c}_{(Q)j} = \hat{b}_{(3)k} \tag{381}$$

be satisfied individually. Using the definitions of these quantities given in (375), (376) and (377), we find from (379) that only terms with

$$R + Q = 3 \tag{382}$$

can pair non-trivially. Substituting this relation into (380) and (381) then constrains i, j and k to satisfy

$$i + j = k \ . \tag{383}$$

We still must show that the sections $\tilde{A}_{(R)}$, $\tilde{a}_{(S)}$, $\tilde{a}$ and $\hat{\tilde{s}}_{(R)i}$, $\hat{\tilde{s}}_{(Q)j}$, $\hat{\tilde{s}}_{(3)k}$ can be chosen to satisfy (379) and (380) respectively. This can be done as follows. Fix once and for all a generic section $\tilde{\ell} \in H^0(B, \mathcal{O}_B(S+\mathcal{E}))$. Now, each of our $\hat{\tilde{s}}$-type sections is the lift of a section $\tilde{s} \in H^0(B, \mathcal{O}_B(aS+b\mathcal{E}))$ for some integers $a \geq b \geq 0$. We will simply set

$$\tilde{s} = \tilde{\ell}^b \tilde{S}^{a-b}, \tag{384}$$

where $\tilde{S}$ is a fixed section of $\mathcal{O}_B(S)$. Note that (384) is a section of the desired bundle $\mathcal{O}_B(aS+b\mathcal{E})$. Furthermore, these sections clearly satisfy (380). We can make similar choices for the a-type sections, thus satisfying (379) as well. Having established the above, we see that any element of $\hat{m}_{(R)i}\hat{\tilde{s}}_{(R)i}\tilde{A}_{(R)}$, $\hat{c}_{(Q)j}\hat{\tilde{s}}_{(Q)j}\tilde{a}_{(Q)}$ or $\hat{b}_{(3)k}\hat{\tilde{s}}_{(3)k}\tilde{a}$ can be explicitly labeled by an element of $M_{(R)i}$, $C_{(Q)j}$ or $B_{(3)k}$ respectively.

Now, consider the matrix $M_1 \in H^0(X, \mathcal{O}_X(\mathcal{C}_V))$. Any matrix block in M_1 that does not satisfy one or both of (382) and (383) must be a zero entry. On the other hand, any entry in M_1 that satisfies both (382) and (383) is a potentially non-vanishing sub-matrix which we denote by

$$m_{(R)i} \in M_{(R)i} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \ . \tag{385}$$

Note that $m_{(R)i}$ may occur as many different sub-matrices within M_1 with the $(R)i$ subscripts fixing its location. This is most easily understood by simply looking at the result. Using the data from (375), (376) and (377) subject to the constraints (382) and (383), we find

$$\begin{pmatrix} m_{(3)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)1} & m_{(2)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)2} & m_{(2)1} & m_{(2)0} & 0 & m_{(1)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)3} & m_{(2)2} & m_{(2)1} & m_{(2)0} & 0 & m_{(1)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)4} & m_{(2)3} & m_{(2)2} & m_{(2)1} & 0 & 0 & m_{(1)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)5} & 0 & m_{(2)3} & m_{(2)2} & 0 & 0 & 0 & m_{(1)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)6} & 0 & 0 & m_{(2)3} & 0 & 0 & 0 & 0 & m_{(1)0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{28\times 84} \cdot \begin{pmatrix} C_{(0)0} \\ C_{(1)1} \\ C_{(1)2} \\ C_{(1)3} \\ C_{(2)2} \\ C_{(2)3} \\ C_{(2)4} \\ C_{(2)5} \\ C_{(2)6} \\ C_{(3)3} \\ C_{(3)4} \\ C_{(3)5} \\ C_{(3)6} \\ C_{(3)7} \\ C_{(3)8} \\ C_{(3)9} \end{pmatrix}_{84} = \begin{pmatrix} B_{(3)0} \\ B_{(3)1} \\ B_{(3)2} \\ B_{(3)3} \\ B_{(3)4} \\ B_{(3)5} \\ B_{(3)6} \end{pmatrix}_{28} . \tag{386}$$

Of course, each non-zero sub-matrix $m_{(R)i}$ maps a space $B_{(Q)j} = H^0(\mathbb{P}^1, \mathcal{O}(j))$ linearly to a space $B_{(3)k} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ where $R+Q=3$ and $i+j=k$. That is, $m_{(R)i}$ is a $(k+1)\times(j+1)$ matrix. We can emphasize this by extending our notation and writing $m_{(R)i}$

as

$$m_{(R)i\{k+1,\ j+1\}}\ . \tag{387}$$

Using this notation, the matrix M_1 can be written as

$$M_1 = \begin{pmatrix} m_{(3)0\{1,1\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)1\{2,1\}} & m_{(2)0\{2,2\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)2\{3,1\}} & m_{(2)1\{3,2\}} & m_{(2)0\{3,3\}} & 0 & m_{(1)0\{3,3\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)3\{4,1\}} & m_{(2)2\{4,2\}} & m_{(2)1\{4,3\}} & m_{(2)0\{4,4\}} & 0 & m_{(1)0\{4,4\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)4\{5,1\}} & m_{(2)3\{5,2\}} & m_{(2)2,\{5,3\}} & m_{(2)1\{5,4\}} & 0 & 0 & m_{(1)0\{5,5\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)5\{6,1\}} & 0 & m_{(2)3\{6,3\}} & m_{(2)2\{6,4\}} & 0 & 0 & 0 & m_{(1)0\{6,6\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{(3)6\{7,1\}} & 0 & 0 & m_{(2)3\{7,4\}} & 0 & 0 & 0 & 0 & m_{(1)0\{7,7\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{28\times 84}\ . \tag{388}$$

It remains to determine the block matrices $m_{(R)i\{k+1,j+1\}}$ to finish constructing M_1. The method for doing this was presented in detail in Section 6 of [32]. We summarize the results here. A block of dimension $(k+1)\times(j+1)$ is a mapping

$$m_{\{k+1,j+1\}} : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j)) \xrightarrow{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-j))} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))\ , \tag{389}$$

where, for the moment, we have suppressed the subscript $(R)i$. Now, we can write $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j))$ in terms of a symmetrized product of the vector space $\hat{V} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. $\hat{V}$ is a two-dimensional space whose basis we choose to be $\{u, v\}$. In other words,

$$\begin{aligned} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j)) &= \mathrm{Sym}^j(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))) \\ &= \mathrm{Sym}^j(\mathrm{span}\{u, v\}) \\ &= \mathrm{span}\{u^j, u^{j-1}v, \ldots, uv^{j-1}, v^j\}. \end{aligned} \tag{390}$$

Similarly,

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = \mathrm{span}\{u^k, u^{k-1}v, \ldots, uv^{k-1}, v^k\}. \tag{391}$$

Finally, $m_{\{k+1,j+1\}}$ itself lives in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-j))$, which can be written as

$$\begin{aligned} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-j)) &= \mathrm{Sym}^{k-j}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))) \\ &= \mathrm{Sym}^{k-j}(\mathrm{span}\{u, v\}) \\ &= \mathrm{span}\{u^{k-j}, u^{k-j-1}v, \ldots, uv^{k-j-1}, v^{k-j}\}. \end{aligned} \tag{392}$$

It is important to note that the sub-matrix is non-zero only when

$$k \geq j, \tag{393}$$

for otherwise $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-j)) = 0$ and our matrix $m_{\{k+1,j+1\}}$ would vanish identically. This is consistent with the constraint that $i + j = k$ given in (383). Having determined the explicit bases, we can now construct the matrix $m_{\{k+1,j+1\}}$. First, note from (392) that any element of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-j))$ can be written in terms of the basis $\{u, v\}$ as

$$m_{\{k+1,j+1\}} = \phi_1 u^{k-j} + \phi_2 u^{k-j-1} v + \ldots + \phi_{k-j+1} v^{k-j}, \tag{394}$$

where $\phi_p \in \mathbb{C}$ are $k-j+1$ complex moduli. Now, tensor expression (394) into each basis element of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j))$ given in (390). Expanding the result into the basis (391) of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ completely specifies the matrix. The final expression for $m_{\{k+1,j+1\}}$ is the following. First of all, the matrix $m_{\{k+1,j+1\}} = 0$ for $k < j$. For $k \geq j$, the p, q-th matrix element is

$$(m_{\{k+1,j+1\}})_{pq} = \begin{cases} \phi_{p-q+1}, & q \leq p \leq q+k-j \ , \\ 0, & \text{otherwise} \end{cases} \tag{395}$$

for $p = 1, 2, \ldots, k+1$ and $q = 1, 2, \ldots, j+1$. It is important to recall, however, that we have been suppressing the $(R)i$ indices on $m_{(R)i\{k+1,j+1\}}$. Restoring these, the expression (394) becomes

$$m_{(R)i\{k+1,j+1\}} = \phi_1^{[(R)i]} u^{k-j} + \phi_2^{[(R)i]} u^{k-j-1} v + \ldots + \phi_{k-j+1}^{[(R)i]} v^{k-j} \ . \tag{396}$$

That is, the moduli labeling $m_{(R)i}$ in $M_{(R)i}$ are uniquely determined by the $(R)i$ indices. Change either or both of these and the set of moduli changes. Note, using the relation (383), that $m_{(R)i}$ is labeled by the moduli

$$\phi_p^{[(R)i]} \ , \qquad p = 1, \ldots, i+1 \ . \tag{397}$$

Of course, expression (395) for the p, q-th element of the matrix $m_{(R)i\{k+1,j+1\}}$ remains the same, but with the moduli replaced by $\phi_{p-q+1}^{[(R)i]}$. At this point, it would be helpful to present some explicit examples. Let us consider the $m_{(1)0\{3,3\}}$ and $m_{(2)1\{3,2\}}$ sub-matrices of M_1 in (388). Then, it follow from (395) that

$$m_{(1)0\{3,3\}} = \begin{pmatrix} \phi_1^{[(1)0]} & 0 & 0 \\ 0 & \phi_1^{[(1)0]} & 0 \\ 0 & 0 & \phi_1^{[(1)0]} \end{pmatrix}, \qquad m_{(2)1\{3,2\}} = \begin{pmatrix} \phi_1^{[(2)1]} & 0 \\ \phi_2^{[(2)1]} & \phi_1^{[(2)1]} \\ 0 & \phi_2^{[(2)1]} \end{pmatrix} . \tag{398}$$

Indeed, expression (395), inserted into each sub-matrix $m_{(R)i\{k+1,j+1\}}$ in (388), completes the construction of the matrix M_1. The general result is unenlightening and will not be presented here.

It is important, however, to compute the generic value of the rank of M_1 To do this, we begin by setting all moduli to zero except those in the $m_{(1)0}$, $m_{(2)0}$ and $m_{(3)0}$ sub-blocks of (388). Note from (398) that

$$m_{(1)0\{3,3\}} = \phi_1^{[(1)0]} \mathbb{1}_{3,3} \ , \tag{399}$$

where $1\!\!1_{3,3}$ is the 3×3 identity matrix. In fact, all $m_{(1)0}$, $m_{(2)0}$ and $m_{(3)0}$ sub-matrices are proportional to $\phi_1^{[(1)0]}$, $\phi_1^{[(2)0]}$ and $\phi_1^{[(3)0]}$ moduli respectively times the unit matrix of the appropriate size. If we then set

$$\phi_1^{[(1)0]} = \phi_1^{[(2)0]} = \phi_1^{[(3)0]} \equiv \phi, \tag{400}$$

we find the matrix M_1 given in (281). For any $\phi \neq 0$, this matrix has

$$\text{rk}(M_1) = 28. \tag{401}$$

One can also consider the complete expression for M_1, randomly initialize the values of its moduli and numerically compute the rank. We have done this for a large number of random initial sets with the caveat, to be explained below, that we insist that all moduli be non-zero. Generically, we recover the result (401). This justifies the important assumption in Subsection 8.2 that we can choose $\text{rk}(M_1) = 28$, thus simplifying the calculation of $h^0(D, W|_D)$ to expression (283).

Similarly, we can construct the matrix M_2, which we recall from (258) is the mapping

$$H^1(X, W(-\mathcal{C}_V))_{180} \xrightarrow{M_2} H^1(X, W)_{91} \ . \tag{402}$$

The result, in block-form, is

$$M_2 = \left(\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)3\{4,1\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)4\{5,1\}} & m_{(6)3\{5,2\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)5\{6,1\}} & m_{(6)4\{6,2\}} & m_{(6)3\{6,3\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)6\{7,1\}} & m_{(6)5\{7,2\}} & m_{(6)4\{7,3\}} & m_{(6)3\{7,4\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\
0 & 0 & 0 & 0 & m_{(6)7\{8,1\}} & m_{(6)6\{8,2\}} & m_{(6)5\{8,3\}} & m_{(6)4\{8,4\}} & m_{(6)3\{8,5\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)8\{9,1\}} & m_{(6)7\{9,2\}} & m_{(6)6\{9,3\}} & m_{(6)5\{9,4\}} & m_{(6)4\{9,5\}} & m_{(6)3\{9,6\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)9\{10,1\}} & m_{(6)8\{10,2\}} & m_{(6)7\{10,3\}} & m_{(6)6\{10,4\}} & m_{(6)5\{10,5\}} & m_{(6)4\{10,6\}} & m_{(6)3\{10,7\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)10\{11,1\}} & m_{(6)9\{11,2\}} & m_{(6)8\{11,3\}} & m_{(6)7\{11,4\}} & m_{(6)6\{11,5\}} & m_{(6)5\{11,6\}} & m_{(6)4\{11,7\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)11\{12,1\}} & m_{(6)10\{12,2\}} & m_{(6)9\{12,3\}} & m_{(6)8\{12,4\}} & m_{(6)7\{12,5\}} & m_{(6)6\{12,6\}} & m_{(6)5\{12,7\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{(6)12\{13,1\}} & m_{(6)11\{13,2\}} & m_{(6)10\{13,3\}} & m_{(6)9\{13,4\}} & m_{(6)8\{13,5\}} & m_{(6)7\{13,6\}} & m_{(6)6\{13,7\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{(4)1\{4,3\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{(4)2\{5,3\}} & m_{(4)1\{5,4\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{(4)3\{6,3\}} & m_{(4)2\{6,4\}} & m_{(4)1\{6,5\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots m_{(4)4\{7,3\}} & m_{(4)3\{7,4\}} & m_{(4)2\{7,5\}} & m_{(4)1\{7,6\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{(4)5\{8,3\}} & m_{(4)4\{8,4\}} & m_{(4)3\{8,5\}} & m_{(4)2\{8,6\}} & m_{(4)1\{8,7\}} & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{(4)6\{9,3\}} & m_{(4)5\{9,4\}} & m_{(4)4\{9,5\}} & m_{(4)3\{9,6\}} & m_{(4)2\{9,7\}} & m_{(4)1\{9,8\}} & 0 & 0 & 0 & 0 & 0 \\
m_{(4)7\{10,3\}} & m_{(4)6\{10,4\}} & m_{(4)5\{10,5\}} & m_{(4)4\{10,6\}} & m_{(4)3\{10,7\}} & m_{(4)2\{10,8\}} & m_{(4)1\{10,9\}} & 0 & 0 & 0 & 0 \\
m_{(4)8\{11,3\}} & m_{(4)7\{11,4\}} & m_{(4)6\{11,5\}} & m_{(4)5\{11,6\}} & m_{(4)4\{11,7\}} & m_{(4)3\{11,8\}} & m_{(4)2\{11,9\}} & m_{(4)\{11,10\}} & 0 & 0 & 0 \\
m_{(4)9\{12,3\}} & m_{(4)8\{12,4\}} & m_{(4)7\{12,5\}} & m_{(4)6\{12,6\}} & m_{(4)5\{12,7\}} & m_{(4)4\{12,8\}} & m_{(4)3\{12,9\}} & m_{(4)2\{12,10\}} & m_{(4)1\{12,11\}} & 0 & 0 \\
0 & m_{(4)9\{13,4\}} & m_{(4)8\{13,5\}} & m_{(4)7\{13,6\}} & m_{(4)6\{13,7\}} & m_{(4)5\{13,8\}} & m_{(4)4\{13,9\}} & m_{(4)3\{13,10\}} & m_{(4)2\{13,11\}} & m_{(4)1\{13,12\}} & 0
\end{matrix}\right)_{91\times 180} . \tag{403}$$

As explicit examples of sub-matrices in M_2, we present the following two which we will use in our analyses below. They are

$$m_{(4)1\{4,3\}} \begin{pmatrix} \phi_1^{[(4)1]} & 0 & 0 \\ \phi_2^{[(4)1]} & \phi_1^{[(4)1]} & 0 \\ 0 & \phi_2^{[(4)1]} & \phi_1^{[(4)1]} \\ 0 & 0 & \phi_2^{[(4)1]} \end{pmatrix}, \qquad m_{(4)2\{5,3\}} = \begin{pmatrix} \phi_1^{[(4)2]} & 0 & 0 \\ \phi_2^{[(4)2]} & \phi_1^{[(4)2]} & 0 \\ \phi_3^{[(4)2]} & \phi_2^{[(4)2]} & \phi_1^{[(4)2]} \\ 0 & \phi_3^{[(4)2]} & \phi_2^{[(4)2]} \\ 0 & 0 & \phi_3^{[(4)2]} \end{pmatrix}. \tag{404}$$

We have computed all other $m_{(R)i\{k+1,j+1\}}$ sub-blocks in (403) but, of course, will not present them here. This completes the construction of the matrix M_2.

E The Rank of M_2

It is not enlightening to display M_2 in its full form in terms of the moduli. However, it is important for us to compute its rank. To do this, we begin by randomly selecting the values of all moduli assuming, however, that each is non-zero. We then numerically compute the rank of M_2. This process is continued for a large number of different random, but non-zero, initializations. The results of an explicit numerical calculation involving 100,000 random integer initializations between 1 and 3 of the moduli are shown in Figure 2. The horizontal axis indicates the ranks of M_2 found in the survey, while the vertical axis gives the number of occurrences. We see that the rank of 85 by far dominates over any other possibilities. It follows that at generic points in moduli space

$$\text{rk}(M_2) = 85. \tag{405}$$

This is, in fact, the maximal possible rank, as can be seen by examining (403) and noting that 6 out of the 91 rows have all zero entries.

Importantly, however, we notice that there are isolated initializations of the moduli for which the rank of M_2 jumps to values smaller than 85. This phenomenon is clearly seen in Figure 2 where $\text{rk}(M_2)$ is shown to attain all integer values between 79 and 84, in addition to its generic value of 85. It is clear from the low statistics of these other values, that they occur at non-generic points in the vector bundle moduli space. In this set of 100,000 integer randomizations, we have not seen any ranks lower than 79. However, as we increase the number of randomizations we expect to see smaller values of $\text{rk}(M_2)$. Statistically, this is expected. For example, with a smaller set of only 1,000 randomizations, only 2 values of $\text{rk}(M_2)$ were observed.

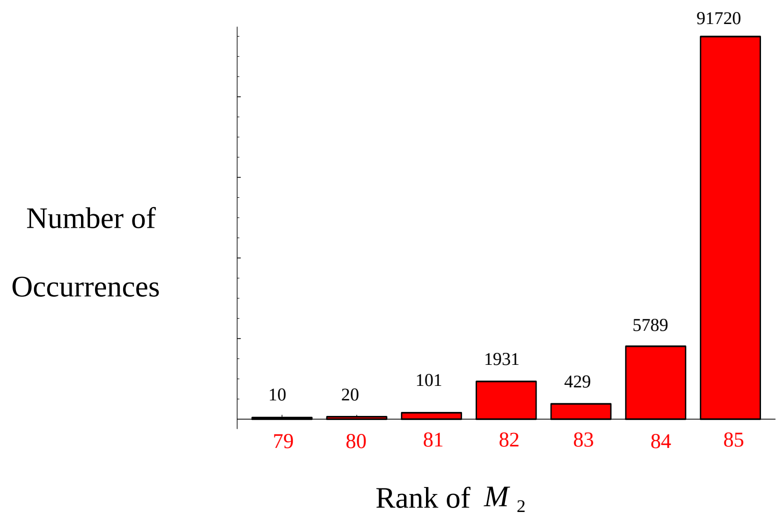

Figure 2: In 100,000 random initializations of the matrix M_2 of integers valued between 1 and 3, the numbers of occurrences of the various values of $\mathrm{rk}(M_2)$ are plotted. We see that the generic value 85 dominates by far.

Clearly, $\mathrm{rk}(M_2)$ jumps to values less than 85 on non-generic points in the moduli space. It is, therefore, rather difficult and unenlightening to search for such points numerically. We have presented the analysis in Figure 2 to clearly demonstrate that the rank of M_2 can take different values in different regions of the vector bundle moduli space. Let us now take a more systematic and analytic approach to this phenomenon of the jumping of the rank. We see from (403) that there are two clusters of non-zero sub-matrices, namely, a triangle cluster consisting of $m_{(6)i}$ sub-matrices and another consisting of $m_{(4)i}$ sub-matrices. Let us leave the sub-blocks $m_{(6)i}$ untouched and proceed to consecutively set the sub-blocks $m_{(4)i}$ to zero. Let us first set the block $m_{(4)1}$, given explicitly in (404), to zero. That is, we set the two moduli

$$\phi_1^{[(4)1]} = \phi_2^{[(4)1]} = 0 \ . \qquad (406)$$

Now compute the rank of M_2 numerically, initializing the remaining moduli to have random, but non-zero, values. Generically, we find

$$\mathrm{rk}(M_2) = 82 \ . \qquad (407)$$

Continuing in this manner, we can set both blocks $m_{(4)1}$ and $m_{(4)2}$ to zero. That is, in addition to (406), take

$$\phi_1^{[(4)2]} = \phi_2^{[(4)2]} = \phi_3^{[(4)2]} = 0 \ . \qquad (408)$$

Again, numerically computing $\mathrm{rk}(M_2)$ for arbitrary, non-zero values of the remaining moduli, we find that

$$\text{rk}(M_2) = 79 \tag{409}$$

generically. And so on. In the table below, we present the generic rank of M_2 evaluated for specific blocks of moduli set to zero.

Block $m_{(4)i}$ set to 0	Generic rank of M_2
none	85
$i = 1$	82
$i = 1, 2$	79
$i = 1, 2, 3$	70
$i = 1, 2, 3, 4$	61
$i = 1, 2, 3, 4, 5$	53
$i = 1, 2, 3, 4, 5, 6$	46
$i = 1, 2, 3, 4, 5, 6, 7$	40
$i = 1, 2, 3, 4, 5, 6, 7, 8$	35
$i = 1, 2, 3, 4, 5, 6, 7, 8, 9$	28

(410)

We see that this procedure stops at rank 28 when we have set all of the triangular cluster of $m_{(4)i}$ sub-matrices to zero while still keeping the $m_{(6)i}$ generic. For reasons to be discussed at the end of this section, we choose not to consider non-generic values of $m_{(6)i}$. Nevertheless, one sees from (410) that we have achieved a wide range of values for $\text{rk}(M_2)$. We have attempted to enlarge this range by modifying our approach and setting only some, but not all, of the moduli within each $m_{(4)i}$ block to zero. Unfortunately, we did not achieve any new, intermediate values for $\text{rk}(M_2)$ beyond those already found.

Not surprisingly, these results differ from those obtained in the purely numerical approach leading to Figure 2. They complement each other in two ways. First, the results in (410) clearly indicate that values of $\text{rk}(M_2)$ much smaller than 79 are attained. On the other hand, the results in Figure 2 imply that, given enough statistics, one would expect $\text{rk}(M_2)$ to attain all integer values from 85 all the way down to 28, not simply the non-consecutive results given in (410). However, we have not definitively proven this.

We can analyze this jumping phenomenon in the following way. There are, as shown in subsection 8.3, 223 vector bundle moduli in our specific example. These parametrize the full moduli space. Of these, there are 139 moduli occurring in M_2, constituting a 139 complex-dimensional subspace $\mathcal{M}$. This number is easily counted from the block form of M_2 given in (403) and expression (395). As one moves in $\mathcal{M}$, the generic value of $\text{rk}(M_2)$ is 85.

However, as one touches certain sub-spaces of $\mathcal{M}$ of co-dimension one or higher, the rank of M_2 drops. At the various intersections of these sub-spaces, that is, at sub-spaces of even higher co-dimension, the rank may drop further. In the example (410), the sub-spaces, in particular, are the coordinate planes in $\mathcal{M}$ where some moduli are set to zero. To be specific, let us consider a particular subspace of $\mathcal{M}$, with three axes corresponding respectively to $\phi^{[(4)1]}_{p=1,2}$, $\phi^{[(4)2]}_{q=1,2,3}$ and $\phi^{[(4)3]}_{r=1,2,3,4}$. In the bulk of this space, $\text{rk}(M_2) = 85$ generically. If we hit the plane $\phi^{[(4)1]}_{p=1,2} = 0$, (410) tells us that $\text{rk}(M_2) = 82$. If we hit the intersection of the planes $\phi^{[(4)1]}_{p=1,2} = 0$ and $\phi^{[(4)2]}_{p=1,2,3} = 0$, then the rank drops to 79. And so on. We present this plot in Figure 3, indicating the various ranks as we restrict to various intersections of the planes within this region of moduli space. We conclude that the rank of M_2 is highly sensitive to where one evaluates it within the moduli space $\mathcal{M}$.

There is one final issue that must be addressed. Recall that the spectral cover $\mathcal{C}_V$ must be irreducible to ensure the stability of V. Now, $\mathcal{C}_V$ is a section of

$$H^0(X, \mathcal{O}_X(\mathcal{C}_V)) = H^0(X, \mathcal{O}_X(5\sigma + \pi^*\eta)) \ . \tag{411}$$

We can decompose this vector space into cohomology groups on $\mathbb{P}^1$ exactly as was done in the explicit construction of the matrix M_1. This was carried out in Appendix D and presented in (377). For convenience, we remind the reader that

$$H^0(X, \mathcal{O}_X(\mathcal{C}_V)) = \bigoplus_{R=1,2,3,4,6} \bigoplus_{i=R-3}^{3R-3} \hat{m}_{(R)i}(\hat{s}_{(R)i} - 3\pi^*\mathcal{E})A_{(R)}, \tag{412}$$

with

$$\begin{aligned}
&\hat{m}_{(R)i} = \pi^* m_{(R)i}, \quad \hat{s}_{(R)i} = \pi^* s_{(R)i}, \quad A_{(R)i} = 5\sigma + \pi^*((6-R)c_1(T\mathbb{F}_1)) \\
&m_{(R)i} = \beta^* M_{(R)i}, \quad s_{(R)i} = Rc_1(T\mathbb{F}_1) - i\mathcal{E} \\
&M_{(R)i} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \ .
\end{aligned} \tag{413}$$

It turns out that a sufficient criterion for the irreducibility of $\mathcal{C}_V$ lies only in the first step in the projection π_* to the base $B = \mathbb{F}_1$. The conditions for the irreducibility of $\mathcal{C}_V$ are found to be

$$\begin{aligned}
&(1) \qquad m_{(1)i} \neq 0, \\
&(2) \qquad m_{(6)i} \text{ generic},
\end{aligned} \tag{414}$$

where $m_{(R)i} \in M_{(R)i}$ are the sub-matrices defined in (385). Therefore, in our matrices M_1 and M_2, whose constituent blocks are the $m_{(R)i}$ sub-matrices, we must make sure that the

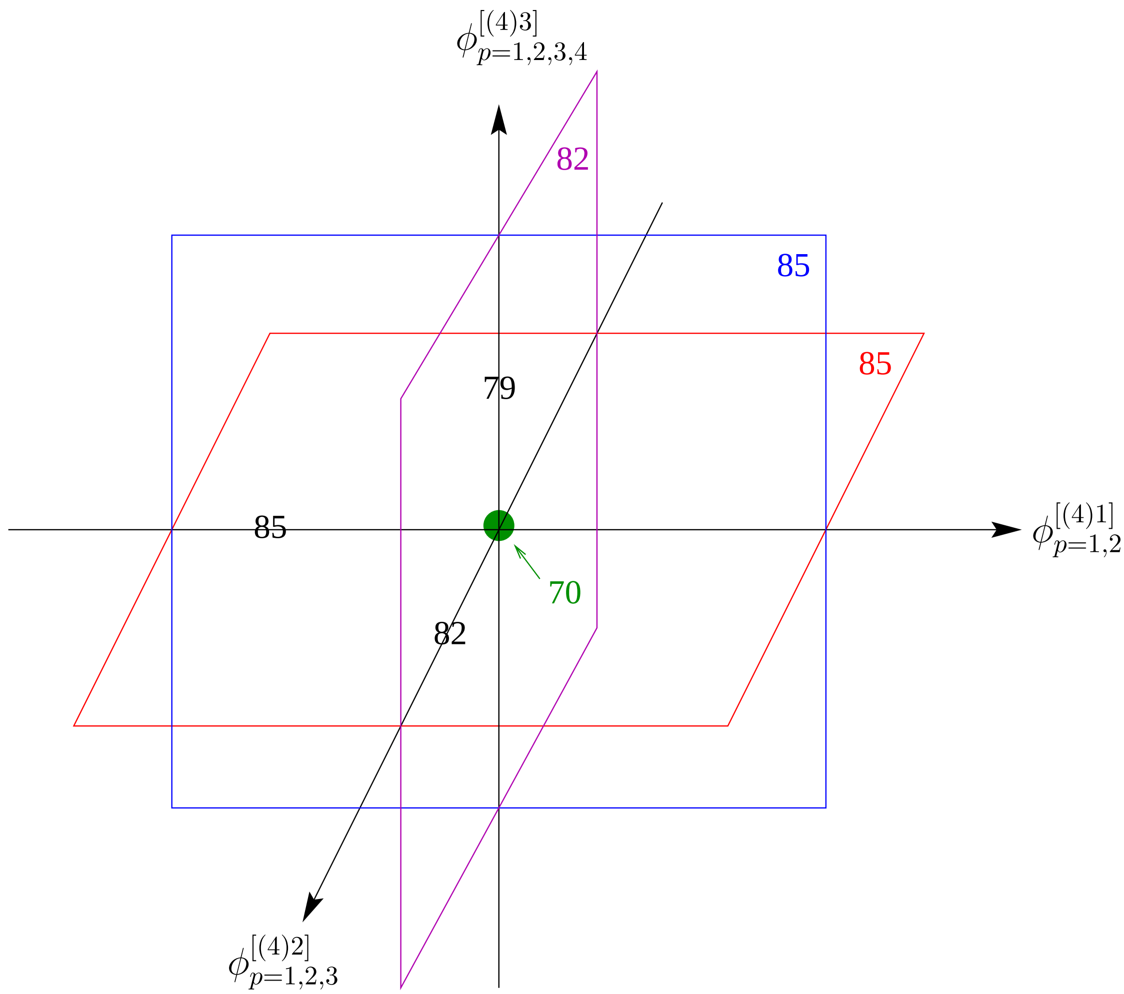

Figure 3: A subspace of the moduli space $\mathcal{M}$ of ϕ's spanned by $\phi^{[(4)1]}_{p=1,2}$, $\phi^{[(4)2]}_{q=1,2,3}$ and $\phi^{[(4)3]}_{r=1,2,3,4}$. Generically, in the bulk, the rank of M_2 is 85, its maximal value. As we restrict to various planes and intersections thereof, we are confining ourselves to special sub-spaces of co-dimension one or higher. In these subspaces, the rank of M_2 can drop dramatically.

two conditions (414) are satisfied when making our choices of the moduli. Now, M_2 does not depend on $m_{(1)i}$ and we have been careful to set $m_{(6)i}$ to generic non-zero values in the above discussions. Furthermore, M_1 does not depend on $m_{(6)i}$ and in our choice of the maximal rank of 28, we have always made sure that $m_{(1)0}$, the only $m_{(1)i}$ sub-block occurring in M_1, is non-zero. Therefore, (414) is indeed satisfied in our choices and the spectral cover $\mathcal{C}_V$ remains irreducible throughout.

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