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The Spectra of Heterotic Standard Model Vacua

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Abstract

A formalism for determining the massless spectrum of a class of realistic heterotic string vacua is presented. These vacua, which consist of $SU(5)$ holomorphic bundles on torus-fibered Calabi-Yau threefolds with fundamental group $\mathbb{Z}_2$, lead to low energy theories with standard model gauge group $(SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$ and three families of quarks and leptons. A methodology for determining the sheaf cohomology of these bundles and the representation of $\mathbb{Z}_2$ on each cohomology group is given. Combining these results with the action of a $\mathbb{Z}_2$ Wilson line, we compute, tabulate and discuss the massless spectrum.

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1 Introduction

The early discussions of realistic vacua in heterotic superstring theory were within the context of the "standard embedding" [1] of the spin connection into the gauge connection. Said differently, these vacua always involve a holomorphic E_8 vector bundle, V, which is induced by the tangent bundle TX of the smooth Calabi-Yau threefold X. Although leading to interesting low energy physics, this approach suffers from the fact that it is highly constrained, the tangent bundle being only one out of an enormous number of possible holomorphic bundles V. One consequence of this constraint is the fact that all heterotic vacua based on the standard embedding require the spontaneous breaking of E_8 to E_6, which is then further broken by Wilson lines. Although E_6 is a possible grand unified group, other groups, such as $SU(5)$ or $Spin(10)$, are simple and more compelling given recent experimental data. Equally significant is that, in the standard embedding, the low energy spectrum and couplings are completely determined by the cohomology of the tangent bundle TX. This seriously constrains these quantities, and it has been difficult to find realistic models in this context.

A technical breakthrough in this regard was presented in [2, 3, 4], where it was shown how to construct a large class of stable, holomorphic vector bundles on simply connected elliptically fibered Calabi-Yau threefolds where $V \neq TX$. Such bundles admit connections satisfying the hermitian Yang-Mills equations. This work was extended in [5]-[13], and it was shown that these bundles can lead to heterotic string vacua with a wide range of low energy gauge groups, including $SU(5)$ and $Spin(10)$. Many of the physical properties of these vacua have been studied, including supersymmetry breaking [14, 15], the moduli space of the vector bundle [16]-[19], and, in the strongly coupled case, the associated M5-brane moduli space [20], small instanton phase transitions [21]-[24], non-perturbative superpotentials [16, 25, 26, 27], and fluxes [28]-[34]. More recently, it was shown how to compute the sheaf cohomology of V and its tensor products, thus determining the complete particle physics spectrum [35, 36]. An important conclusion of these papers is that the spectrum depends on the region of vector bundle moduli space in which it is evaluated. Although constant for generic moduli, the spectrum can jump dramatically on subspaces of co-dimension one or higher always containing, however, three families of quarks and leptons. These vacua also underlie the theory of brane universes [6]-[12] and ekpyrotic and Big Crunch/Big Bang cosmology [37]-[40]. The major drawback of these vacua is that the compactification manifold is simply connected. It follows that these are all GUT theories which cannot be broken to the standard model with Wilson lines [41]-[47]. Although many of these vacua contain Higgs multiplets whose vacuum expectation values could induce symmetry breaking, it would be simpler and more natural if Wilson lines could be introduced.

This was accomplished in [48]-[51], where stable holomorphic vector bundles with structure group $SU(5)$ were constructed over torus-fibered Calabi-Yau threefolds with fundamental group $\pi_1(X) = \mathbb{Z}_2$. These heterotic vacua lead, using a $\mathbb{Z}_2$ Wilson line, to low energy theories that are anomaly free, have three families of quarks/leptons and the gauge group $(SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$. This work was extended to vector bundles with structure group $SU(4)$ on torus-fibered Calabi-Yau threefolds with $\pi_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_2$ in [52]-[54] and $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$ in [55]. Although very promising, it is essential that one now compute the exact spectrum and couplings in these standard model vacua. In this paper, we take a major step in this direction by computing the particle spectrum for the vacua in [48]-[51].

This is accomplished as follows. In [48]-[51], X is the quotient $X = \tilde{X}/\mathbb{Z}_2$, where $\tilde{X}$ is a simply connected Calabi-Yau threefold. Denote by $\tilde{V}$ the pull-back of V to $\tilde{V}$. To find the particle spectrum, one must first compute the sheaf cohomology of $\tilde{V}$ and its tensor products. This is a non-trivial task involving various techniques in cohomological algebra and algebraic geometry. In this paper, we present a systematic approach to such computations, and determine all relevant cohomology groups in our theory. The next step is to find the explicit representations of $\mathbb{Z}_2$ in each of these spaces. We give a precise methodology for accomplishing this. This approach is then used to determine each of the requisite $\mathbb{Z}_2$ representations. The above information, in conjunction with the action of the $\mathbb{Z}_2$ Wilson line, can be utilized to find all group multiplets that are invariant under $\mathbb{Z}_2$, as well as their multiplicities. When constructing the quotient Calabi-Yau threefold $X = \tilde{X}/\mathbb{Z}_2$, these invariant multiplets descend to X and form the massless particle physics spectrum. Using these techniques, we compute and tabulate the spectrum.

Specifically, we do the following. In Section 2, we present a general formalism for describing $G(\subset E_8)$-bundles, Wilson lines and the massless spectrum associated with non-simply connected Calabi-Yau threefolds X with $\pi_1(X) = F$. It is shown that determining this spectrum requires the computation of specific sheaf cohomologies on the covering Calabi-Yau threefold $\tilde{X}$, as well as the action of F on these groups. This formalism is illustrated for several values of F, including $F = \mathbb{Z}_2$. Section 3 is devoted to a brief review of the results in [48]-[51]. Specifically, we discuss the construction of torus-fibered Calabi-Yau threefolds X with fundamental group $F = \mathbb{Z}_2$. It is shown how to construct stable, holomorphic bundles V with structure group $SU(5)$ on X. These arise from $\mathbb{Z}_2$ invariant bundles $\tilde{V}$ on $\tilde{X}$ and satisfy the basic phenomenological constraints of particle physics. Computing the massless spectrum of this theory requires determining the sheaf cohomology of $\tilde{V}$ and its tensor products. A general method for doing this is presented in Section 4 and used to compute the relevant cohomology groups in our theory. Section 5 is devoted to finding the explicit representations of $\mathbb{Z}_2$ on these cohomology groups. Combining the results of Section 5 with the $F = \mathbb{Z}_2$ example in Section 2, the massless spectrum of our theory is computed, tabulated and discussed in Section 5. Finally, in the Appendix we present some useful mathematical facts used throughout the paper.

2 The Spectra of Heterotic Compactifications with Wilson Lines

A vacuum in weakly coupled heterotic string theory is specified by a pair $(X, \overline{\mathcal{V}})$, where X is a Calabi-Yau threefold and $\overline{\mathcal{V}}$ is a stable $E_8 \times E_8$ holomorphic principal bundle on X satisfying the Green-Schwarz anomaly cancellation condition [56]

$$c_2(\overline{\mathcal{V}}) = c_2(TX) \ . \tag{1}$$

Note that specifying the $E_8 \times E_8$ bundle $\overline{\mathcal{V}}$ is the same as giving two E_8 bundles $\mathcal{V}$ and $\mathcal{V}''$. The anomaly cancellation condition can be written as

$$c_2(\mathcal{V}) + c_2(\mathcal{V}'') = c_2(TX) \ . \tag{2}$$

In this work, we will always take $\mathcal{V}''$ to be trivial. Then, condition (2) becomes

$$c_2(\mathcal{V}) = c_2(TX) \ . \tag{3}$$

In heterotic M-theory compactifications, this condition is relaxed to

$$c_2(\mathcal{V}) + [C] = c_2(TX) \ , \tag{4}$$

where $\mathcal{V}$ is a stable holomorphic E_8 principal bundle in the observable sector and $[C]$ is the class of some effective curve $C \subset X$ on which M5-branes wrap.

The particle spectrum of this compactification consists [1] of zero-modes of the ten-dimensional Dirac operator

$$\not{D} : \Gamma(\mathrm{ad}\mathcal{V} \otimes S^+_{10}) \to \Gamma(\mathrm{ad}\mathcal{V} \otimes S^-_{10}) \ . \tag{5}$$

Here $\mathrm{ad}\mathcal{V}$ is the rank-248 vector bundle associated to $\mathcal{V}$ by the adjoint representation of E_8, $S^{\pm}_{10}$ are the bundles of positive and negative chirality spinors in 10-dimensions, and Γ

denotes global sections of a bundle over the 10-dimensional space $\mathbb{R}^4 \times X$. (Note that we can consider $\mathrm{ad}\mathcal{V}$ to be a bundle on $\mathbb{R}^4 \times X$ by simply pulling it back from X).

The 10-dimensional spinors decompose in terms of their (Minkowski) $\mathbb{R}^4$ and (internal) X components as

$$S_{10}^{+} = (S_4^{+} \otimes S_6^{+}) \oplus (S_4^{-} \otimes S_6^{-}). \tag{6}$$

The internal spinors, on the Calabi-Yau threefold X, can be identified with the $(0,q)$ forms $\mathcal{A}^{0,q}$ on X, with even/odd q corresponding to positive/negative chirality:

$$S_6^{+} \simeq \mathcal{A}^{0,0} \oplus \mathcal{A}^{0,2}, \qquad S_6^{-} \simeq \mathcal{A}^{0,1} \oplus \mathcal{A}^{0,3}. \tag{7}$$

In terms of this identification, the Dirac operator becomes $\not{D} = \overline{\partial} + \overline{\partial}^{*} + \not{D}_4$ coupled to $\mathrm{ad}\mathcal{V}$, where $\overline{\partial}$ is the Dolbeault operator on X, and $\not{D}_4$ is the Dirac operator on flat $\mathbb{R}^4$. Putting these facts together, we find that the spectrum is

$$\ker(\not{D}) \simeq \left(\bigoplus_{q=0,2} H^q(X, \mathrm{ad}\mathcal{V}) \otimes \mathbf{S}_4^{+} \right) \oplus \left(\bigoplus_{q=1,3} H^q(X, \mathrm{ad}\mathcal{V}) \otimes \mathbf{S}_4^{-} \right), \tag{8}$$

where $\mathbf{S}_4^{\pm}$ denote the constant sections of the bundle $S_4^{\pm}$ on $\mathbb{R}^4$. The positive chirality particles are those which multiply $\mathbf{S}_4^{+}$, so they are given by (a basis of)

$$\bigoplus_{q=0,2} H^q(X, \mathrm{ad}\mathcal{V}). \tag{9}$$

Their negative chirality anti-particles are similarly given by a basis of

$$\bigoplus_{q=1,3} H^q(X, \mathrm{ad}\mathcal{V}). \tag{10}$$

By Serre duality, this is the dual space to (9), as it should be by CPT. Recall that, for each charged particle, CPT predicts the existence of an anti-particle of opposite charge. The annihilation of a particle with its anti-particle can be interpreted as a natural pairing. Hence, we can interpret the space of anti-particles as the dual of the space of particles. In order to describe the complete spectrum, we will in this work calculate

$$Spec := \bigoplus_{q=0,1} H^q(X, \mathrm{ad}\mathcal{V}). \tag{11}$$

Then, $\ker(\not{D})$ is obtained by adding the duals to *Spec*.

In practice, the E_8 bundle $\mathcal{V}$ is often associated to some stable G-bundle V on X, where $G \subset E_8$ is some subgroup, e.g., $G = SU(n)$ for $n = 3, 4$ or 5 [1]:

$$\mathcal{V} = V \overset{G}{\times} E_8 \ . \tag{12}$$

The resulting compactification then has a low energy gauge group

$$H = Z_{E_8}(G) \ , \tag{13}$$

the commutant of G in E_8. The decomposition of the 248-dimensional representation adE_8 under the product $G \times H$ then gives an associated decomposition for ad$\mathcal{V}$ and the Dirac-operator zero-modes. For example, we can take V to be an $SU(3)$ bundle, or equivalently, a rank 3 vector bundle with trivial determinant. The usual embedding of $G = SU(3)$ into E_8 has commutant $H = E_6$. The decompostion of adE_8 into irreducible representations of $SU(3) \times E_6$ involves four terms

$$248 = (1, 78) \oplus (8, 1) \oplus (3, 27) \oplus (\overline{3}, \overline{27}) \ . \tag{14}$$

Here, 8 and 78 are the adjoints of $SU(3)$ and E_6 respectively, 3 is the fundamental representation of $SU(3)$, and 27, $\overline{27}$ are the smallest representations of E_6. For the zero-modes we get:

$$Spec = \big(H^0(X, \mathcal{O}_X) \otimes 78\big) \oplus \big(H^1(X, \mathrm{ad}V) \otimes \mathbb{1}\big) \oplus \big(H^1(X, V) \otimes 27\big) \oplus \big(H^1(X, V^*) \otimes \overline{27}\big) \ . \tag{15}$$

Here we think of V as a rank 3 vector bundle on X associated to the principal $SU(3)$ bundle by the fundamental representation, V^* is its dual vector bundle, ad$V$ is the rank-8 vector bundle of traceless endomorphisms of V, and $\mathcal{O}_X$ is the trivial rank-1 bundle on X. Note that the stability of V and the Calabi-Yau property of X guarantee that for each of the associated bundles $(\mathcal{O}_X, \mathrm{ad}V, V, V^*)$, the cohomology can be non-zero for either $q = 0$ or $q = 1$ but not both, as indicated in (15).

As another example, we consider the usual embedding of $G = SU(5)$ into E_8. The commutant is $H = SU(5)$ and the $SU(5)_G \times SU(5)_H$-decomposition is

$$248 = (1, 24) \oplus (24, 1) \oplus (10, 5) \oplus (\overline{10}, \overline{5}) \oplus (5, \overline{10}) \oplus (\overline{5}, 10) \ . \tag{16}$$

[1]Since all of our bundles are holomorphic, the relevant structure groups are actually $G = SL(n, \mathbb{C})$. However, to conform to the usual physics notation, we will throughout this paper refer to these groups as $G = SU(n)$.

The zero-modes are

$$
\begin{aligned}
Spec \; = \; & \big(H^0(X,\mathcal{O}_X)\otimes 24\big) \oplus \big(H^1(X,\mathrm{ad}V)\otimes \mathbb{1}\big) \oplus \big(H^1(X,\wedge^2 V)\otimes 5\big) \oplus \big(H^1(X,\wedge^2 V^*)\otimes \overline{5}\big) \\
& \oplus \big(H^1(X,V)\otimes \overline{10}\big) \oplus \big(H^1(X,V^*)\otimes 10\big) \; . \qquad (17)
\end{aligned}
$$

More generally, for $G \subset E_8$ with commutant H, we write

$$
\mathrm{ad}E_8 = \bigoplus_i U_i \otimes R_i \; , \qquad (18)
$$

where U_i runs over irreducible representations of G, and R_i are corresponding representations of H. Using this decomposition of the representation $\mathrm{ad}E_8$ on each fiber of the E_8 bundle defined in (12), we find the decomposition

$$
\mathrm{ad}\mathcal{V} = \bigoplus_i U_i(V) \otimes R_i \; , \qquad (19)
$$

where $U_i(V)$ are the vector bundles associated to the G-bundle V via the representations U_i of G.

Next we want to see how these results are modified by Wilson lines. Let $F \subset H$ be a finite subgroup which acts on a Calabi-Yau threefold $\tilde{X}$ freely with a Calabi-Yau quotient $X = \tilde{X}/F$. The G-bundle V and the E_8-bundle $\mathcal{V} = V \overset{G}{\times} E_8$ on X pull back to a G-bundle $\tilde{V} = p^*V$ and an E_8-bundle $\tilde{\mathcal{V}} = p^*\mathcal{V} = \tilde{V} \overset{G}{\times} E_8$ on $\tilde{X}$, where $p : \tilde{X} \to X$ is the covering map. The action of F on $\tilde{X}$ lifts to actions, denoted ρ, on $\tilde{V}$, $\tilde{\mathcal{V}}$, hence on their cohomologies. The cohomology group computed on X is precisely the $\rho(F)$-invariant part of the cohomology on $\tilde{X}$

$$
H^q(X, \mathrm{ad}\mathcal{V}) = H^q(\tilde{X}, \mathrm{ad}\tilde{\mathcal{V}})^{\rho(F)} \; . \qquad (20)
$$

The Wilson line W is the flat H-bundle on X induced from the F-cover $p : \tilde{X} \to X$ via the given embedding of F in H:

$$
W := \tilde{X} \overset{F}{\times} H \; . \qquad (21)
$$

The $(G \times H)$-bundle $V \oplus W$ induces another E_8-bundle on X:

$$
\mathcal{V}' = (V \oplus W) \overset{(G\times H)}{\times} E_8 \; . \qquad (22)
$$

Our goal in this work is to study the particle spectrum and other properties of the heterotic vacuum given by compactification on $(X, \mathcal{V}')$. Since the structure group of $\mathcal{V}'$ can be reduced to $G \times F$ (but not to G), we see in analogy with (13) that this vacuum has low energy gauge group

$$
S := Z_H(F) = Z_{E_8}(G \times F) \; . \qquad (23)
$$

We will work primarily with a particular class of geometric examples which is reviewed in Section 2. In the remainder of the present section we will describe the general approach. This is based on the observation that, when pulled backed to $\tilde{X}$, the two bundles $\mathcal{V}$, $\mathcal{V}'$ coincide:

$$p^*\mathcal{V}' \simeq p^*\mathcal{V} = \tilde{\mathcal{V}} \ . \tag{24}$$

This is because the finite structure group F of the Wilson line W is killed in the passage from X to $\tilde{X}$. Another way to describe this is to note that there are two actions ρ, ρ' of F on $\tilde{\mathcal{V}}$, both lifting the given F action on $\tilde{X}$. The quotient by ρ gives $\mathcal{V}$, and the quotient by ρ' gives $\mathcal{V}'$. The analogue of (20) is:

$$H^q(X, \mathrm{ad}\mathcal{V}') = H^q(\tilde{X}, \mathrm{ad}\tilde{\mathcal{V}})^{\rho'(F)} \ . \tag{25}$$

We can write the decomposition (19) on $\tilde{X}$:

$$\mathrm{ad}\tilde{\mathcal{V}} = \bigoplus_i U_i(\tilde{V}) \otimes R_i \tag{26}$$

and use formulas (20), (25) to descend to X. The ρ action of F acts only on the associated vector bundles $U_i(\tilde{V})$, hence on their cohomology, so:

$$H^q(X, \mathrm{ad}\mathcal{V}) = \bigoplus_i H^q(\tilde{X}, U_i(\tilde{V}))^{\rho(F)} \otimes R_i \ . \tag{27}$$

The ρ' action of F is a combination of the ρ action on the $U_i(\tilde{V})$ with the action of $F \subset H$ on the R_i:

$$H^q(X, \mathrm{ad}\mathcal{V}') = \bigoplus_i \left(H^q(\tilde{X}, U_i(\tilde{V})) \otimes R_i\right)^{\rho'(F)} \ . \tag{28}$$

Recall that $H^q(X, \mathrm{ad}\mathcal{V})$ and its decomposition (27) carry an action of H (which is the natural action on R_i in (27)), but only the subgroup $S \subset H$ acts on $H^q(X, \mathrm{ad}\mathcal{V}')$ and its decomposition (28). To make the latter more explicit, we decompose each H-representation R_i in terms of the irreducible F-representations A_j:

$$R_i = \bigoplus_j (A_j \otimes B_{ij}), \quad B_{ij} := \mathrm{Hom}_F(A_j, R_i) \ . \tag{29}$$

Our formula (28) for the particle spectrum then becomes

$$H^q(X, \mathrm{ad}\mathcal{V}') = \bigoplus_{i,j} (H^q(\tilde{X}, U_i(\tilde{V})) \otimes A_j)^{\rho'(F)} \otimes B_{ij} \ . \tag{30}$$

Here each B_{ij} carries a representation of the low energy gauge group S, which occurs in $H^q(X, \mathrm{ad}\mathcal{V}')$ with multiplicity m_{ij} equal to the dimension of the space of F-invariants in

$H^q(\tilde{X}, U_i(\tilde{V})) \otimes A_j$. Note that the S-representation B_{ij} is often not irreducible. Rather, we should think of B_{ij} as a block of irreducible S-representations, each of which corresponds to some particle. All the particles in a given block B_{ij} occur in the spectrum with the same multiplicity m_{ij}.

We can summarize our procedure so far as follows. The input involves

- a structure group $G \subset E_8$,
- a finite subgroup F of the commutant $H = Z_{E_8}(G)$,
- a free action of F on a Calabi-Yau threefold $\tilde{X}$ with Calabi-Yau quotient $X = \tilde{X}/F$, and
- a G-bundle V on X satisfying the anomaly cancellation condition (4).

These data determine a Wilson line W on X (as in (21)) and a heterotic vacuum $(X, \mathcal{V}')$ where $\mathcal{V}'$ combines the G-bundle V with the Wilson line W, as in (22). The low energy gauge group of this vacuum is the subgroup $S \subset H$ given in (23). The particle spectrum is determined as follows:

- Decompose $\mathrm{ad}E_8$ as in (18) in terms of irreducible G-representations U_i and corresponding H-representations R_i.
- Decompose each R_i as in (29) in terms of irreducible F-representations A_j and corresponding blocks of irreducible S-representations B_{ij}.
- Most of the work then goes into computing the cohomology groups $H^q(\tilde{X}, U_i(\tilde{V}))$ of the associated vector bundles on $\tilde{X}$, and the action of F on these cohomologies. The multiplicity m_{ij} of the irreducible F-representation A_j^* in $H^q(\tilde{X}, U_i(\tilde{V}))$ is the multiplicity of all particles from block B_{ij} in the particle spectrum of $(X, \mathcal{V}')$.

We illustrate the general procedure in two cases. First consider $G = SU(3)$, $H = E_6$. As we saw in (14), the U_i are 1, 8, 3 and $\overline{3}$, and the corresponding R_i are 78, 1, 27 and $\overline{27}$. Now $H = E_6$ has a maximal subgroup

$$H_0 = SU(3)_C \times SU(3)_L \times SU(3)_R \, , \tag{31}$$

where we can think of C, L, R as standing for color, left, right. We can, for example, take $F = F(n, \hat{n}) = \mathbb{Z}_n \times \mathbb{Z}_{\hat{n}}$ whose two generators are mapped to H_0 as

$$\mathbb{1}_C \times \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{pmatrix}_L \times \mathbb{1}_R \, , \qquad \mathbb{1}_C \times \mathbb{1}_L \times \begin{pmatrix} \hat{\alpha} & & \\ & \hat{\alpha} & \\ & & \hat{\alpha}^{-2} \end{pmatrix}_R \, , \tag{32}$$

where α and $\hat{\alpha}$ are roots of unity of orders n and $\hat{n}$ respectively. Another possibility is to work with F_0, the diagonal subgroup $\mathbb{Z}_n$ in $F(n,n)$, with generator

$$\mathbb{1}_C \times \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{pmatrix}_L \times \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{pmatrix}_R . \tag{33}$$

Either F (with $n, \hat{n} > 1$) or F_0 (with $n > 1$) break E_6 to

$$S = SU(3)_C \times \left(\frac{SU(2) \times U(1)}{\mathbb{Z}_2}\right)_L \times \left(\frac{SU(2) \times U(1)}{\mathbb{Z}_2}\right)_R . \tag{34}$$

In this case, it is easier to first decompose each R_i under H_0, and then to further decompose each H_0 component under F and S. Under H_0 we have:

$$\begin{aligned} 78 &= (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (3,3,3) \oplus (\overline{3},\overline{3},\overline{3}) \\ 1 &= (1,1,1) \\ 27 &= (3,\overline{3},1) \oplus (1,3,\overline{3}) \oplus (\overline{3},1,3) \\ \overline{27} &= (\overline{3},3,1) \oplus (1,\overline{3},3) \oplus (3,1,\overline{3}) , \end{aligned} \tag{35}$$

where (a,b,c) is shorthand for the H_0-representation $a_C \otimes b_L \otimes c_R$. When we further decompose under S, the color representations are unchanged, while the 3 of L or R breaks as $2_1 \oplus 1_{-2}$, and the adjoint 8 breaks as $3_0 \oplus 1_0 \oplus 2_3 \oplus 2_{-3}$. (Here b_w denotes the b-dimensional representation of $SU(2)$, on which $U(1)$ acts with weight w. This representation of $SU(2) \times U(1)$ factors through $(SU(2) \times U(1))/\mathbb{Z}_2$ if and only if the integers b and w have opposite parity.) So the $(8,1,1)$ of H_0 becomes $(8,1,1)_{0,0}$ of S, while the $(1,8,1)$ becomes $(1,3,1)_{0,0} \oplus (1,1,1)_{0,0} \oplus (1,2,1)_{3,0} \oplus (1,2,1)_{-3,0}$. The two subscripts give the weights of the two $U(1)$'s in S. The same subscripts taken modulo n and $\hat{n}$ give the weights of $F(n,\hat{n})$, so they determine the representation A_j. We tabulate the results in Table 1. In that table, the only reducible block is B_{00}. However, if we replace F by its subgroup F_0, many of the A_j coalesce, resulting in many reducible B_{ij}'s.

For our second example we consider $G = SU(5)$, so $H = SU(5)$ and the decomposition of adE_8 is given in (16). The finite group F is $\mathbb{Z}_2$, where the generator is embedded in $H = SU(5)$ diagonally with eigenvalues $(1,1,1,-1,-1)$. This breaks H to the standard model group $S = (SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$. In Table 2, we use $(a,b)_w$ to denote the product of an a-dimensional representation of $SU(3)$ with a b-dimensional representation of $SU(2)$, where $U(1)$ acts with weight $w = 3Y$. The corresponding representation A_j of F depends only on the parity of w.

U_i	$H^q(\tilde{X}, U_i(\tilde{V}))$	R_i	A_j	B_{ij}
1	$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$	78	$0, 0$	$(8,1,1) \oplus (1,3,1) \oplus (1,1,3) \oplus 2 \times (1,1,1)$
			$3, 0$	$(1,2,1)$
			$-3, 0$	$(1,2,1)$
			$0, 3$	$(1,1,2)$
			$0, -3$	$(1,1,2)$
			$1, 1$	$(3,2,2)$
			$-2, -2$	$(3,1,1)$
			$-1, -1$	$(\overline{3},2,2)$
			$2, 2$	$(\overline{3},1,1)$
			$1, -2$	$(3,2,1)$
			$-2, 1$	$(3,1,2)$
			$-1, 2$	$(\overline{3},2,1)$
			$2, -1$	$(\overline{3},1,2)$
8	$H^1(\tilde{X}, \mathrm{ad}\tilde{V})$	1	$0, 0$	$(1,1,1)$
3	$H^1(\tilde{X}, \tilde{V})$	27	$-1, 0$	$(3,2,1)$
			$2, 0$	$(3,1,1)$
			$1, -1$	$(1,2,2)$
			$-2, -1$	$(1,1,2)$
			$1, 2$	$(1,2,1)$
			$-2, 2$	$(1,1,1)$
			$0, 1$	$(\overline{3},1,2)$
			$0, -2$	$(\overline{3},1,1)$
$\overline{3}$	$H^1(\tilde{X}, \tilde{V^*})$	$\overline{27}$	$-1, 0$	$(\overline{3},2,1)$
			$2, 0$	$(\overline{3},1,1)$
			$1, -1$	$(1,2,2)$
			$-2, -1$	$(1,1,2)$
			$1, 2$	$(1,2,1)$
			$-2, 2$	$(1,1,1)$
			$0, 1$	$(3,1,2)$
			$0, -2$	$(3,1,1)$

Table 1: The decomposition of $H^q(X, \mathrm{ad}\mathcal{V}')$ where $G = SU(3)$ and $F = \mathbb{Z}_n \times \mathbb{Z}_{\hat{n}}$.

U_i	$H^q(\tilde{X}, U_i(\tilde{V}))$	R_i	A_j	B_{ij}
1	$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$	24	0	$(8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0$
			1	$(3,2)_{-5} \oplus (\overline{3},2)_5$
24	$H^1(\tilde{X}, \mathrm{ad}\tilde{V})$	1	0	$(1,1)_0$
10	$H^1(\tilde{X}, \wedge^2\tilde{V})$	5	0	$(3,1)_{-2}$
			1	$(1,2)_3$
$\overline{10}$	$H^1(\tilde{X}, \wedge^2\tilde{V}^*)$	$\overline{5}$	0	$(\overline{3},1)_2$
			1	$(1,2)_{-3}$
5	$H^1(\tilde{X}, \tilde{V})$	$\overline{10}$	0	$(3,1)_4 \oplus (1,1)_{-6}$
			1	$(\overline{3},2)_{-1}$
$\overline{5}$	$H^1(\tilde{X}, \tilde{V}^*)$	10	0	$(\overline{3},1)_{-4} \oplus (1,1)_6$
			1	$(3,2)_1$

Table 2: The decomposition of $H^q(X, \mathrm{ad}\mathcal{V}')$ where $G = SU(5)$ and $F = \mathbb{Z}_2$. The A_j correspond to characters of the $\mathbb{Z}_2$ action on R_i. The a, b in $(a,b)_w$ are the representations of $SU(3)_C$ and $SU(2)_L$ respectively, whereas $w = 3Y$.

3 Standard Model Bundles

In this section we recall the standard model bundles constructed in [48, 49, 50]. We need a quadruple $(\tilde{X}, A, \tau, \tilde{V})$ satisfying:

- $(\mathbb{Z}_{\mathbf{2}})$ $\tilde{X}$ is a smooth Calabi-Yau 3-fold and $\tau : \tilde{X} \to \tilde{X}$ is a freely acting involution.
  A is a fixed ample line bundle (Kähler structure) on $\tilde{X}$.
- **(S)** $\tilde{V}$ is an A-stable vector bundle of rank five on $\tilde{X}$ with structure group $G = SU(5)$.
- **(I)** $\tilde{V}$ is τ-invariant.
- **(C1)** $c_1(\tilde{V}) = 0$.
- **(C2)** $c_2(\tilde{X}) - c_2(\tilde{V})$ is effective.
- **(C3)** $c_3(\tilde{V}) = 12$. (36)

The involution τ generates a subgroup $\mathbb{Z}_2 = F \subset H = Z_{E_8}(SU(5)) = SU(5)$. The quotient $X := \tilde{X}/F$ is another Calabi-Yau threefold, and invariance of $\tilde{V}$ allows us to identify it with the pullback of a stable $SU(5)$ bundle V on X, as in Section 2. This produces a heterotic M-theory vacuum $(X, \mathcal{V}')$ with particle spectrum as given in Table 2 of Section 2.

3.1 Rational Elliptic Surfaces and Their Products

The simply connected threefold $\tilde{X}$ is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^2 \times (\mathbb{P}^2)'$ of two hypersurfaces of multidegrees $(1,3,0)$ and $(1,0,3)$ respectively. This is a Calabi-Yau, by adjunction, and it has two elliptic fibrations. These threefolds were first studied by Schoen [57]. Choose projective coordinates: $[t_0 : t_1]$ on $\mathbb{P}^1$; $z = [z_0 : z_1 : z_2]$ on $\mathbb{P}^2$; and $z' = [z'_0 : z'_1 : z'_2]$ on $(\mathbb{P}^2)'$. The two hypersurfaces can be written:

$$t_0 f_0(z) - t_1 f_1(z) = 0 \tag{37}$$

$$t_0 f'_0(z') - t_1 f'_1(z') = 0 , \tag{38}$$

where f_0, f_1, f'_0, f'_1 are homogeneous cubic polynomials. Since equation (37) does not involve z', it defines a hypersurface $B \subset \mathbb{P}^1 \times \mathbb{P}^2$. Similarly equation (38) defines a hypersurface $B' \subset \mathbb{P}^1 \times (\mathbb{P}^2)'$. The surfaces B, B' are called rational elliptic surfaces, or (inaccurately) dP_9's. The projections of these surfaces to $\mathbb{P}^1$ are elliptic fibrations:

$$\beta : B \to \mathbb{P}^1, \qquad \beta' : B' \to \mathbb{P}^1 . \tag{39}$$

The original threefold $\tilde{X}$ comes with the two projections

$$\pi : \tilde{X} \to B', \qquad \pi' : \tilde{X} \to B \tag{40}$$

which are again elliptic fibrations. In fact, $\tilde{X}$ is the fiber product

$$\tilde{X} = B \times_{\mathbb{P}^1} B' , \tag{41}$$

meaning that a point of $\tilde{X}$ is uniquely specified by a pair of points $b \in B$, $b' \in B'$ with $\beta(b) = \beta'(b') \in \mathbb{P}^1$.

The opposite projection $\nu : B \to \mathbb{P}^2$ is birational, exhibiting B as the blowup of $\mathbb{P}^2$ at the 9 points A_i, $i = 1, \ldots, 9$ where $f_0(z) = f_1(z) = 0$, and similarly for B'. (This is the origin of the "dP_9" name – but these surfaces are not del Pezzos.) It follows that $H^2(B, \mathbb{Z}) = Pic(B)$ has rank 10. An orthogonal basis consists of the class $\ell := \nu^* \mathcal{O}_{\mathbb{P}^2}(1)$ together with the 9 exceptional classes $e_1, \ldots, e_9$. The only non-zero intersection numbers on B are $\ell^2 = 1$, $e_i^2 = -1$, $i = 1, \ldots, 9$. The class $f := \beta^{-1}(\text{point})$ of an elliptic fiber is given by $f = 3\ell - \sum_{i=1}^{9} e_i$. There is an analogous basis $\ell', e'_1, \ldots, e'_9$ on B'. The rank of $H^2(\tilde{X}, \mathbb{Z})$ is therefore 19, with basis $\pi^*\ell' = (\pi')^*\ell, \pi^* e'_1, \ldots, \pi^* e'_9, (\pi')^* e_1, \ldots, (\pi')^* e_9$.

3.2 Special Rational Elliptic Surfaces

In order to obtain the involution τ on $\tilde{X}$, and also in order to have invariant bundles $\tilde{V}$ on $\tilde{X}$ satisfying the required conditions, the rational elliptic surfaces B, B' need to be specialized to a particular subfamily. This can be specified as follows.

Let $\Gamma_1 \subset \mathbb{P}^2$ be a nodal cubic with a node A_8. Choose four generic points on Γ_1 and label them A_1, A_2, A_3, A_7. Let $\Gamma \subset \mathbb{P}^2$ be the unique smooth cubic which passes through A_1, A_2, A_3, A_7, A_8 and is tangent to the lines $\langle A_7 A_i \rangle$ for $i = 1, 2, 3$ and 8. Consider the pencil of cubics spanned by Γ_1 and Γ. All cubics in this pencil pass through A_1, A_2, A_3, A_7, A_8 and are tangent to Γ at A_8. Let A_4, A_5, A_6 be the remaining three base points, and let B denote the blow-up of $\mathbb{P}^2$ at the points A_i, $i = 1, 2, \ldots, 8$ and the point A_9 which is infinitesimally near A_8 and corresponds to the line $\langle A_7 A_8 \rangle$.

The pencil becomes the anti-canonical map $\beta : B \to \mathbb{P}^1$ which is an elliptic fibration with a section. The map β has two reducible fibers $f_i = n_i \cup o_i$, $i = 1, 2$ of type I_2. We denote by e_i, $i = 1, \ldots, 7$ and e_9 the exceptional divisors corresponding to A_i, $i = 1, \ldots, 7$ and A_9, and by e_8 the reducible divisor $e_9 + n_1$. The divisors e_i together with the pullback ℓ of a class of a line from $\mathbb{P}^2$ form a standard basis in $H^2(B, \mathbb{Z})$.

The surface B has an involution α_B which is uniquely characterized by the properties: $\beta \circ \alpha_B = \tau_{\mathbb{P}^1} \circ \beta$, where $\tau_{\mathbb{P}^1}$ is the involution $t_0 \to t_0, t_1 \to -t_1$ on $\mathbb{P}^1$, and α_B fixes the proper transform of Γ pointwise. Note that $\tau_{\mathbb{P}^1}$ leaves two points in $\mathbb{P}^1$ fixed, which we call 0 and ∞. Furthermore, α_B acts as $(-1)_B$ when restricted to the fiber $f_\infty = \beta^{-1}(\infty)$ and, hence, leaves four points fixed in f_∞.

Choosing $e_9 := e$ as the zero section of β, we can interpret any other section ξ as an automorphism $t_\xi : B \to B$ which acts along the fibers of β. The automorphism $\tau_B = t_{e_1} \circ \alpha_B$ is again an involution of B which commutes with β, induces the same involution on $\mathbb{P}^1$ as α_B, and has four isolated fixed points sitting on the same fiber f_∞ of β.

The special rational elliptic surfaces form a four dimensional irreducible family. Their geometry was the subject of [48]. The structure of a special rational elliptic surface B is shown in Figure 1 and the action of τ_B on $H^2(B, \mathbb{Z})$ is summarized in Table 3.

3.3 Building $\tilde{X}, \tau$ and A

Choose two special rational elliptic surfaces $\beta : B \to \mathbb{P}^1$ and $\beta' : B' \to \mathbb{P}^1$ in $\tilde{X}$ so that the discriminant loci of β and β' in $\mathbb{P}^1$ are disjoint, α_B and $\alpha_{B'}$ induce the same involution on $\mathbb{P}^1$, and the fixed loci of τ_B and $\tau_{B'}$ sit over different points 0 and ∞ in $\mathbb{P}^1$. The fiber

	τ_B^*
e_1	e_9
$e_j (j = 2, 3)$	$f - e_j + e_1 + e_9$
$e_i (i = 4, 5, 6)$	$f - l + e_i + e_1 + e_7 + e_9$
e_7	$l - e_2 - e_3$
e_8	$f - l + e_1 + e_7 + e_8 + e_9$
$e_9 = e$	e_1
l	$2f + 2(e_1 + e_9) - (e_2 + e_3) + e_7$
$f = 3l - \sum_{i=1}^{9} e_i$	f

Table 3: The action of τ_B on $H^2(B, \mathbb{Z})$.

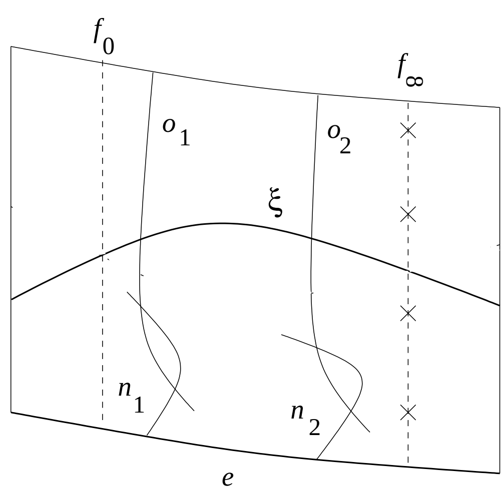

Figure 1: A special rational elliptic surface B. It has 8 I_1 singular fibers. In addition, there are 2 I_2 fibers $f_1 = n_1 \cup o_1$ and $f_2 = n_2 \cup o_2$. Under the involution $\tau_B = t_\xi \circ \alpha_B$, there are 4 fixed points, which we have marked, on the fiber f_∞.

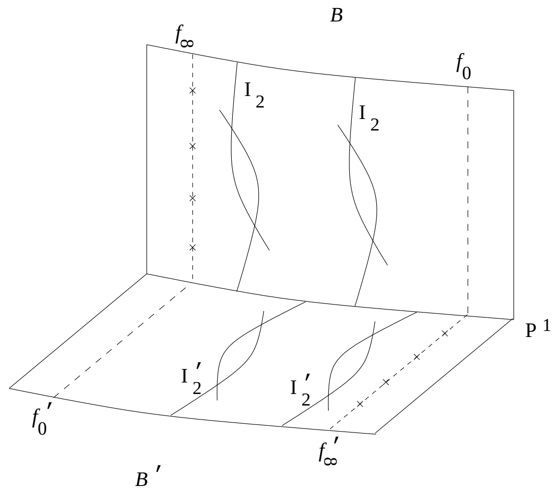

Figure 2: The Calabi-Yau threefold $\tilde{X}$ is constructed as the fiber product over $\mathbb{P}^1$ of two non-generic dP_9 surfaces B and B'. We have matched the fibers f_0 and f_∞ of B with the fibers f'_∞ and f'_0 of B' respectively. The image points in $\mathbb{P}^1$ of these fibers, namely 0 and ∞ for B and $0'$ and ∞' for B', are identified as $0 = \infty'$ and $\infty = 0'$.

product $\tilde{X} = B \times_{\mathbb{P}^1} B'$ is a smooth Calabi-Yau threefold which is elliptic and has a freely acting involution $\tau := \tau_B \times_{\mathbb{P}^1} \tau_{B'}$ and another (non-free) involution $\alpha_X := \alpha_B \times_{\mathbb{P}^1} \alpha_{B'}$. For concreteness we fix the elliptic fibration of $\tilde{X}$ to be the projection $\pi : \tilde{X} \to B'$ to B'. The structure of $\tilde{X}$ is shown in Figure 2.

The stability of the bundle $\tilde{V}$ which we describe below is with respect to a particular choice of Kähler class A. If A_0 is any Kähler class on $\tilde{X}$, h' a Kähler class on B', and $n \gg 0$, the class of $A = A_0 + n\pi^* h'$ will be Kähler on $\tilde{X}$. The specific value that was found in [49] to satisfy all the requirements was given by $h' = 193f' + 144e'_1 + 168(e'_9 + e'_4 - e'_5)$.

3.4 The Construction of V

The construction of the $SU(5)$ bundle V on $X := \tilde{X}/F$ is equivalent to the construction of an $SU(5)$ bundle $\tilde{V}$ on $\tilde{X}$ together with an action of the involution τ on $\tilde{V}$. The construction of $\tilde{V}$ in [49] employs a combination of two techniques: extensions and the spectral construction.

The rank 5 bundle $\tilde{V}$ is constructed as an extension

$$0 \to V_2 \to \tilde{V} \to V_3 \to 0 \tag{42}$$

involving two simpler bundles V_2, V_3, of ranks 2 and 3 respectively. Given the V_i, we can immediately construct their direct sum $\tilde{V}_0 = V_2 \oplus V_3$, which is the trivial extension. In terms of an open cover $\{U_\alpha\}$ and $i \times i$ transition matrices $\{g_{i\alpha\beta}\}$ for each V_i, the transition matrices for $\tilde{V}_0$ are

$$g_{0\alpha\beta} = \begin{pmatrix} g_{2\alpha\beta} & 0 \\ 0 & g_{3\alpha\beta} \end{pmatrix} . \tag{43}$$

A general extension $\tilde{V}$ is a rank 5 bundle containing V_2 as a subbundle with quotient V_3, but V_3 cannot be realized as a subbundle of $\tilde{V}$ unless $\tilde{V}$ is the trivial extension $\tilde{V}_0$. The transition matrices for such an extension must be of the form:

$$g_{\alpha\beta} = \begin{pmatrix} g_{2\alpha\beta} & h_{\alpha\beta} \\ 0 & g_{3\alpha\beta} \end{pmatrix} . \tag{44}$$

In order for these $g_{\alpha\beta}$ to define a vector bundle, the upper right corner $h_{\alpha\beta}$ must satisfy a cocycle condition. Working this out shows that the set of isomorphism classes of extensions is described by the sheaf cohomology group:

$$\mathrm{Ext}^1_{\tilde{X}}(V_3, V_2) := H^1(\tilde{X}, V_3^* \otimes V_2) . \tag{45}$$

The direct sum $\tilde{V}_0 = V_2 \oplus V_3$ corresponds to the 0 element of this extension group. Our standard model bundle $\tilde{V}$ turns out to correspond to a non-trivial extension $[\tilde{V}] \in \mathrm{Ext}^1_{\tilde{X}}(V_3, V_2)$. In order for $\tilde{V}$ to be τ-invariant, we require first that V_2 and V_3 be τ-invariant, so we can choose an action of τ on V_2 and V_3. This induces an action of τ on $\mathrm{Ext}^1_{\tilde{X}}(V_3, V_2)$. In order for $\tilde{V}$ to be τ-invariant, we require that the extension class $[\tilde{V}]$ be τ-invariant.

3.5 The Construction of the V_i

The construction of the bundles V_i, $i = 2, 3$, involves the **spectral construction** or **Fourier-Mukai transform** [2, 3, 4]. The Fourier-Mukai transform is a self-equivalence of the derived category $D^b(\tilde{X})$ of coherent sheaves on $\tilde{X}$

$$\begin{aligned} FM : \quad & D^b(\tilde{X}) \to D^b(\tilde{X}) \\ & \mathcal{F} \to Rp_{1*}(p_2^*\mathcal{F} \overset{L}{\otimes} \mathcal{P}) . \end{aligned} \tag{46}$$

Here, p_1, p_2 are the projections of the fiber product $\tilde{X} \times_{B'} \tilde{X}$ to the two $\tilde{X}$ factors, Rp_{1*} is the right derived functor of p_{1*}, $\mathcal{P}$ is the Poincaré sheaf on $\tilde{X} \times_{B'} \tilde{X}$, and $\overset{L}{\otimes}$ is the left derived functor of $\otimes$. If V_i is a rank i vector bundle on $\tilde{X}$ which is semistable and of degree 0 on each elliptic fiber f of $\pi : \tilde{X} \to B'$, then $FM^{-1}(V_i)$ is a rank 1 sheaf N_{Σ_i} supported on a divisor $\Sigma_i \subset \tilde{X}$ which is finite of degree i over the base B'. In other words, Σ_i intersects each

$$
\begin{tikzcd}
 & & V_i \arrow[d] & & \\
 & & \tilde{X} \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & & \\
W_i \arrow[r] & B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] & L_i \arrow[l] \; . \\
 & & \mathbb{P}^1 & &
\end{tikzcd}
$$

Figure 3: The structure of the vector bundles V_i, $i = 2, 3$.

elliptic fiber f in i points. In case Σ_i is smooth, N_{Σ_i} is actually a line bundle on Σ_i. The spectral construction starts with (Σ_i, N_{Σ_i}) and recovers the bundle V_i as the Fourier-Mukai transform. When Σ_i is irreducible, the resulting bundle V_i is automatically stable.

In our case we do not need the full spectral construction on the threefold $\tilde{X}$. The map $\beta : B \to \mathbb{P}^1$ is an elliptic fibration, so there is a Fourier-Mukai transform FM_B on $D^b(B)$. We will describe below certain curves $C_i \subset B$ and line bundles $N_i \in Pic(C_i)$ for $i = 2, 3$. These determine two bundles $W_i := FM_B(C_i, N_i)$ with $\mathrm{rk}(W_i) = i$. Our desired bundles V_i are then recovered as

$$
V_i = \pi'^* W_i \otimes \pi^* L_i \tag{47}
$$

for appropriate line bundles $L_i \in Pic(B')$. The spectral data on B and on $\tilde{X}$ are related by

$$
\Sigma_i = (\pi')^{-1} C_i = C_i \times_{\mathbb{P}^1} B', \qquad N_{\Sigma_i} = (\pi')^* N_i \otimes \pi^* L_i \ . \tag{48}
$$

This is summarized in Figure 3.

The specific values we take for the C_i, N_i and L_i are as follows. Choose curves $\overline{C}_2, C_3 \subset B$, so that

- $\overline{C}_2 \in |\mathcal{O}_B(2e_9 + 2f)|, \quad C_3 \in |\mathcal{O}_B(3e_9 + 6f)|$,
- $\overline{C}_2$ and C_3 are α_B-invariant,
- $\overline{C}_2$ and C_3 are smooth and irreducible.

Set $C_2 = \overline{C}_2 + f_\infty$ where f_∞ is the smooth fiber of β containing the four fixed points of τ_B. We choose the line bundles $N_2 \in Pic^{3,1}(C_2)$, $N_3 \in Pic^{16}(C_3)$ to transform correctly under the involution $\alpha_B|_{C_i}$:

$$N_i \simeq (\alpha_B|_{C_i})^* N_i \otimes \mathcal{O}_{C_i}(e_1 - e_9 + f), \qquad i = 2, 3 \; . \tag{49}$$

Here $Pic^{3,1}(C_2)$ denotes line bundles of degree 3 on $\overline{C}_2$ and degree 1 on f_∞ [49]. (It is shown in [49] that such N_i do exist.) A useful quantity associated with the bundle W_2 is the degree -1 line bundle $G \in Pic^{-1}(f_\infty)$ on the elliptic curve f_∞, defined as

$$G = N_2|_{f_\infty}(-D), \tag{50}$$

where D is the divisor $D = \overline{C}_2 \cap f_\infty$. This fits into an exact sequence

$$0 \to W_2 \to \overline{W}_2 \to i_{f_\infty *}(G^*) \to 0 \; , \tag{51}$$

where $\overline{W}_2$ is the rank 2 vector bundle associated with the spectral cover $\overline{C}_2$ and spectral line bundle $\overline{N}_2 = N_2 \otimes \mathcal{O}_{\overline{C}_2}$. The Chern characters can be read from Lemma 5.1 of [49]:

$$\begin{aligned} \mathrm{ch}(W_2) &= 2 - f - 3\mathrm{pt}, \quad \mathrm{ch}(\overline{W}_2) = 2 - 2\mathrm{pt}, \\ \mathrm{ch}(W_3) &= 3 + f - 6\mathrm{pt}, \quad\; \mathrm{ch}(G^*) = f + \mathrm{pt}. \end{aligned} \tag{52}$$

Finally, the line bundles L_i on B' are given by

$$\begin{aligned} L_2 &= \mathcal{O}_{B'}(3r') \\ L_3 &= \mathcal{O}_{B'}(-2r') \end{aligned} \tag{53}$$

where

$$r' = e_1' + e_4' - e_5' + e_9' + f' = 3\ell' - 2e_4' - (e_2' + e_3' + e_6' + e_7' + e_8'). \tag{54}$$

Formula (53) holds with the specific choices $N_2 \in Pic^{3,1}(C_2)$, $N_3 \in Pic^{16}(C_3)$ which we made above, and only with those choices. This is why we specify the general solution in [49] to these values.

This completes the specification of the bundles V_i for $i = 2, 3$. It was seen in [49] that τ-invariant extensions $[\tilde{V}] \in \mathrm{Ext}^1_{\tilde{X}}(V_3, V_2)_{(+)}$ exist, and that the bundle $\tilde{V}$ corresponding to a general such $[\tilde{V}]$ has structure group $G = SU(5)$, is stable, is τ-invariant, and satisfies the requirements $(\mathbb{Z}_2, S, I, C1, C2, C3)$ in (36).

3.6 Comments

The reason we did not build $\tilde{V}$ directly by a spectral construction applied to the surface $\Sigma = \Sigma_2 \cup \Sigma_3$ in $\tilde{X}$ (or to the curve $C = C_2 \cup C_3$ in B) is that on singular spectral covers (such as Σ, C), the rank 1 sheaf ($\overline{N}$ or N) can fail to be a line bundle, leading to technical complications. A closely related problem is that it is harder to check the stability of $\tilde{V}$ when the spectral cover is reducible.

Another subtlety is that our C_2 is not finite over $\mathbb{P}^1$. It intersects the generic elliptic fiber in 2 points, but it contains the entire fiber f_∞. We chose N_2 carefully so that our W_2 is still the Fourier-Mukai transform of (C_2, N_2). But in practice it is often easier to work with $\overline{C}_2$, $\overline{N}_2$ and $\overline{W}_2$, and to relate W_2 and $\overline{W}_2$ via (51).

The construction in [49] involves additional degrees of freedom in the form of Hecke transforms applied to the $\tilde{V}$. Later checks, motivated by questions of Mike Douglas, suggest that most or all of these extra degrees of freedom may be illusory. At any rate, we do not use them in the present work.

4 Cohomologies of $U_i(\tilde{V})$

In order to compute the relevant cohomologies on a rational elliptic surface such as B', we need some basic facts about the line bundle $\mathcal{O}_{B'}(r')$ of (54). We claim that the direct image is:

$$\beta'_* \mathcal{O}_{B'}(r') \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \tag{55}$$

or equivalently that

$$\beta'_* \mathcal{O}_{B'}(r' - f') \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \tag{56}$$

Indeed, the left hand side of (56) is a rank 2 bundle on $\mathbb{P}^1$, since $(r' - f') \cdot f' = 2$, so it must be of the form $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some integers a, b. Now $r' - f' = e'_1 + e'_9 + e'_4 - e'_5$ cannot be effective (any effective representative has negative intersection with e'_1, e'_4, e'_9, so must contain all of them), and therefore our integers a, b must be negative. To conclude that $a = b = -1$ as claimed in (56), it suffices to note that $a + b$ is the degree of $\beta'_* \mathcal{O}_{B'}(r' - f')$, which by Groethendieck-Riemann-Roch (GRR) equals -2.

Instead of GRR, we can obtain the same result using a bit of geometry. We saw in (54) that $r' = 3\ell' - (e'_2 + e'_3 + e'_6 + e'_7 + e'_8) - 2e'_4$, so we can identify sections of $\mathcal{O}_{B'}(r')$ with cubic polynomials on $\mathbb{P}^2$ vanishing at A_i for $i = 2, 3, 6, 7, 8$, and vanishing to second order at A_4. The space $H^0(\mathcal{O}_{\mathbb{P}^2}(3\ell))$ of cubics is 10 dimensional, the vanishing at each of the five A_i imposes one linear condition, and vanishing to second order at A_4 imposes 3 more conditions, for a total of 8 conditions. Therefore $2 = 10 - 8 \leq h^0(\mathcal{O}_{B'}(r')) = h^0(\mathbb{P}^1, \beta'_*\mathcal{O}_{B'}(r')) = h^0(\mathcal{O}_{\mathbb{P}^1}(a+1)) + h^0(\mathcal{O}_{\mathbb{P}^1}(b+1))$. Recalling that a, b are negative, this is possible only for $a = b = -1$; so we have found another argument for (55), (56).

It follows from (55) that $H^0(\mathcal{O}_{B'}(r'))$ is 2 dimensional. We let x_0 and x_1 be a basis. We claim that the quotient x_1/x_0 is everywhere defined, so it gives a map

$$\chi : B' \to \mathbb{P}^1_x, \tag{57}$$

and the x_i can be interpreted as homogeneous coordinates on the target $\mathbb{P}^1_x$. Checking that χ is everywhere defined is equivalent to verifying that x_0 and x_1 cannot vanish at the same point. Since $r'^2 = 0$, two divisors in the linear system $|r'|$ cannot intersect each other unless they have a common component. So to conclude, it suffices to check that some (and hence almost all) of these divisors are irreducible. This follows immediately from the geometric model: in fact, the fibers of χ, identified as the pencil of cubics vanishing at the five e'_i and doubly at e'_4, include precisely 8 reducible curves, namely:

$$\begin{array}{lll} K_i^1 \cup K_i^2, \quad K_i^1 = \ell' - e'_5 - e'_i, & K_i^2 = 2\ell' - (e'_2 + e'_3 + e'_6 + e'_7 + e'_8) - e'_5 + e'_i, & i = 2,3,6,7,8 \\ K_j^0 \cup K_j^3, \quad K_j^0 = e'_j, & K_j^3 = 3\ell' - (e'_2 + e'_3 + e'_6 + e'_7 + e'_8) - 2e'_5 - e'_j, & j = 1,4,9. \end{array} \tag{58}$$

The first five curves occur as reducible cubics in $\mathbb{P}^2$, consisting of the line joining A_5 to A_i and the conic through A_5 and the remaining 4 points. The last three consist of cubics which happen to pass through one of the A_j, so their inverse image in B' contains the corresponding e'_j. All other cubics in our system are singular (at A_5) but irreducible. We conclude that χ is indeed everywhere defined, its generic fiber is a $\mathbb{P}^1$, and precisely the 8 fibers listed in (58) split into a pair of $\mathbb{P}^1$'s meeting at one point.

Clearly, the target space $\mathbb{P}^1_x$ of the map χ defined by the line bundle $\mathcal{O}_{B'}(r')$ has nothing to do with the target space $\mathbb{P}^1_t$ of the map β' defined by the line bundle $\mathcal{O}_{B'}(f')$. In fact, we can put these two maps together, to get a map

$$\Delta = (\beta', \chi) : B' \to \mathsf{Q} := \mathbb{P}^1_t \times \mathbb{P}^1_x \tag{59}$$

given by the two pairs of homogeneous coordinates (t_0, t_1), (x_0, x_1).

The product surface Q could be identified with a smooth quardric in $\mathbb{P}^3$ via the embedding $(t_0x_0, t_1x_0, t_0x_1, t_1x_1)$, but we will not use this. The product map Δ is onto Q, and is of degree $f' \cdot r' = 2$; in other words, we have realized the rational elliptic surface B' as a double cover of the quadric surface Q. The fibers of β' are of course the elliptic curves f' which now appear as double covers of $\mathbb{P}^1_x$ branched at 4 points. The general fiber of χ, on the other hand, is isomorphic to a $\mathbb{P}^1$, as is seen by adjunction. It appears as a double cover of $\mathbb{P}^1_t$ branched at 2 points. The branch locus Br_Δ of Δ is therefore a divisor of bidegree $(4, 2)$ in Q .

Line bundles on Q are of the form $\mathcal{O}_\mathsf{Q}(k,l) := pr_t^*\mathcal{O}_{\mathbb{P}^1_t}(l) \otimes pr_x^*\mathcal{O}_{\mathbb{P}^1_x}(k)$, with integers k, l, where pr_t, pr_x are the projections to $\mathbb{P}^1_t$, $\mathbb{P}^1_x$ respectively: $\beta' = pr_t \circ \Delta$, $\chi = pr_x \circ \Delta$. Let us introduce the abbreviation

$$\mathcal{O}_{B'}(k,l) := \Delta^*\mathcal{O}_\mathsf{Q}(k,l) = \mathcal{O}_{B'}(kr' + lf') \tag{60}$$

for the corresponding line bundles on B'. So for example $\mathcal{O}_{B'}(0,1)$ is the anticanonical bundle $K_{B'}^{-1} \simeq \mathcal{O}_{B'}(f')$, $\mathcal{O}_{B'}(1,0)$ is $\mathcal{O}_{B'}(r')$, $\mathcal{O}_{B'}(3,0)$ is our L_2, and $\mathcal{O}_{B'}(-2,0)$ is L_3.

On B' there is a unique involution ι which exchanges the two sheets of B' over Q . Its fixed locus is the ramification divisor $Ram_\Delta \subset B'$. The image $\Delta(Ram_\Delta)$ is of course Br_Δ. Since

$$\Delta^*\mathcal{O}_\mathsf{Q}(Br_\Delta) = \mathcal{O}_{B'}(2Ram_\Delta) \tag{61}$$

and the Picard group of B' has no torsion, we find that:

$$\mathcal{O}_{B'}(Ram_\Delta) \simeq \Delta^*\mathcal{O}_\mathsf{Q}\left(\frac{1}{2}Br_\Delta\right) = \Delta^*\mathcal{O}_\mathsf{Q}(2,1) = \mathcal{O}_{B'}(2,1). \tag{62}$$

For any double cover such as Δ, sections of $\mathcal{O}_{B'}$ can be decomposed into ι-invariants and anti-invariants. This can be written formally as a decomposition of the direct image:

$$\Delta_*\mathcal{O}_{B'} \simeq 1 \cdot \mathcal{O}_\mathsf{Q} \oplus y \cdot \mathcal{O}_\mathsf{Q}\left(-\frac{1}{2}Br_\Delta\right), \tag{63}$$

where $y \in H^0(\mathcal{O}_{B'}(Ram_\Delta))$ is the ι-anti-invariant section characterized up to scalars by its vanishing precisely on Ram_Δ. (This is another special case of GRR). In our case, (62) shows that

$$y \in H^0(\mathcal{O}_{B'}(2,1)), \quad \iota y = -y \tag{64}$$

and

$$\Delta_*\mathcal{O}_{B'} = \mathcal{O}_\mathsf{Q} \oplus y\mathcal{O}_\mathsf{Q}(-2,-1). \tag{65}$$

This can be tensored with the pullback of $\mathcal{O}_\mathsf{Q}(k,l)$, giving the decomposition

$$\Delta_*\mathcal{O}_{B'}(k,l) = \mathcal{O}_\mathsf{Q}(k,l) \oplus y\mathcal{O}_\mathsf{Q}(k-2,l-1) \tag{66}$$

which will be the foundation for our cohomological calculations.

Let $S_x^k := H^0(\mathcal{O}_{\mathbb{P}^1_x}(k))$ denote the $(k+1)$-dimensional vector space of homogeneous polynomials of degree $k \geq 0$ in x_0, x_1, with basis consisting of the monomials $x_0^k, x_0^{k-1}x_1, \ldots, x_1^k$.

We set $S_x^k = 0$ for $k < 0$, and let $(S_x^k)^*$ denote the dual vector space. The cohomology of $\mathbb{P}^1_x$ is given by:

$$H^0(\mathcal{O}_{\mathbb{P}^1_x}(k)) = S_x^k, \quad H^1(\mathcal{O}_{\mathbb{P}^1_x}(k)) \simeq (S_x^{-2-k})^*, \tag{67}$$

where the second formula involves Serre duality and therefore depends on choosing, once and for all, an isomorphism of $K_{\mathbb{P}^1_x}$ with $\mathcal{O}_{\mathbb{P}^1_x}(-2)$. This formula can be applied to the product surface $\mathsf{Q} = \mathbb{P}^1_t \times \mathbb{P}^1_x$, yielding a formula for the direct images (for a general definition of direct image sheaves we refer the reader to the Appendix)

$$R^i pr_{t*}\mathcal{O}_{\mathsf{Q}}(k,l) = H^i(\mathcal{O}_{\mathbb{P}^1_x}(k)) \otimes \mathcal{O}_{\mathbb{P}^1_t}(l) \simeq \begin{cases} S_x^k \otimes \mathcal{O}_{\mathbb{P}^1_t}(l), & i = 0 \\ (S_x^{-2-k})^* \otimes \mathcal{O}_{\mathbb{P}^1_t}(l), & i = 1 \end{cases}, \tag{68}$$

and therefore for the cohomology:

$$\begin{aligned} H^n(\mathcal{O}_{\mathsf{Q}}(k,l)) &= \bigoplus_{i+j=n} H^i(\mathcal{O}_{\mathbb{P}^1_x}(k)) \otimes H^j(\mathcal{O}_{\mathbb{P}^1_t}(l)) \\ &\simeq \begin{cases} S_x^k \otimes S_t^l, & n = 0 \\ (S_x^{-2-k})^* \otimes S_t^l \oplus S_x^k \otimes (S_t^{-2-l})^*, & n = 1 \\ (S_x^{-2-k})^* \otimes (S_t^{-2-l})^* & n = 2. \end{cases} \end{aligned} \tag{69}$$

The power of formula (66) is that it allows us to write down analogous formulas for the much more complicated surface B':

$$\begin{array}{rcccc} \beta'_*\mathcal{O}_{B'}(k,l) & = & S_x^k \otimes \mathcal{O}_{\mathbb{P}^1_t}(l) & \oplus & yS_x^{k-2} \otimes \mathcal{O}_{\mathbb{P}^1_t}(l-1) \\ R^1\beta'_*\mathcal{O}_{B'}(k,l) & \simeq & (S_x^{-2-k})^* \otimes \mathcal{O}_{\mathbb{P}^1_t}(l) & \oplus & y(S_x^{-k})^* \otimes \mathcal{O}_{\mathbb{P}^1_t}(l-1). \end{array} \tag{70}$$

Note that for $k > 0$ only the β'_* term is non-zero, while for $k < 0$ only the $R^1\beta'_*$ term is non-zero. The cohomology on B' can be obtained from (70), or directly from (66):

$$H^n(\mathcal{O}_{B'}(k,l)) = H^n(\mathcal{O}_{\mathsf{Q}}(k,l)) \oplus yH^n(\mathcal{O}_{\mathsf{Q}}(k-2,l-1)), \tag{71}$$

where the individual terms are given in (69).

Explicitly, this formula gives bases for the various cohomology groups on B' consisting of monomials in t_0, t_1, x_0, x_1, y. For example:

$$\begin{array}{ll} H^0(\mathcal{O}_{B'}(0,1)): & t_0, t_1 \\ H^0(\mathcal{O}_{B'}(1,0)): & x_0, x_1 \\ H^0(\mathcal{O}_{B'}(3,0)): & x_0^3, x_0^2x_1, x_0x_1^2, x_1^3 \\ H^0(\mathcal{O}_{B'}(2,1)): & t_0x_0^2, t_0x_0x_1, t_0x_1^2, t_1x_0^2, t_1x_0x_1, t_1x_1^2, y. \end{array} \tag{72}$$

Now, we are ready to calculate the cohomology groups which we need on $\tilde{X}$.

• $\boxed{V_2}$ We have

$$\beta_* \overline{W}_2 = \beta_* W_2 = 0 \tag{73}$$

since these sheaves are torsion-free and vanish at a generic point. We also have $R^1\beta_*\overline{W}_2 = 0$ because it is supported on $\overline{C}_2 \cap e_9$, which is empty. The long exact sequence induced from (51) therefore gives:

$$0 = \beta_* \overline{W}_2 \to \beta_* i_{f_\infty *}(G^*) \to R^1\beta_* W_2 \to R^1\beta_* \overline{W}_2 \to 0, \tag{74}$$

so $R^1\beta_* W_2 = \beta_* i_{f_\infty *}(G^*)$. The Leray spectral sequence for $\pi : \tilde{X} \to B'$ therefore gives:

$$\begin{aligned} H^1(\tilde{X}, V_2) &= H^1(\tilde{X}, \pi'^* W_2 \otimes \pi^* L_2) = H^0(B', R^1\pi_*\pi'^* W_2 \otimes L_2) \\ &= H^0(B', \beta'^* R^1\beta_* W_2 \otimes L_2) = H^0(f_\infty, G^*) \otimes H^0(f_0', L_2). \end{aligned} \tag{75}$$

Note that $h^0(f_\infty, G^*) = 1$, $h^0(f_0', L_2) = 6$, hence $h^1(\tilde{X}, V_2) = 6$.

• $\boxed{V_3}$ We again have $\beta_* W_3 = 0$, so for $i = 0, 1$:

$$H^i(\tilde{X}, V_3) = H^0(B', \beta'^* R^i\beta_* W_3 \otimes L_3) = H^0(\mathbb{P}^1, R^i\beta_* W_3 \otimes \beta'_* L_3) = 0, \tag{76}$$

where we have used that $\beta'_* L_3 = 0$, which holds since $L_3 \cdot f' = -4 < 0$.

• $\boxed{\tilde{V}}$ The long exact sequence induced from (42) gives:

$$0 = H^0(\tilde{X}, V_3) \to H^1(\tilde{X}, V_2) \to H^1(\tilde{X}, \tilde{V}) \to H^1(\tilde{X}, V_3) = 0, \tag{77}$$

so $H^1(\tilde{X}, \tilde{V}) = H^1(\tilde{X}, V_2) = H^0(f_\infty, G^*) \otimes H^0(f_0', L_2)$ by (75).

• $\boxed{\wedge^2 V_2}$ From (52) we know that $\wedge^2 W_2 = c_1(W_2) = -f$. But $\pi'^* \mathcal{O}_B(-f) \simeq \pi^* \mathcal{O}_{B'}(-f')$, since both pull back from the same sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$. Therefore,

$$\wedge^2 V_2 = \pi'^* \wedge^2 W_2 \otimes \pi^*(L_2^{\otimes 2}) \simeq \pi^* \mathcal{O}_{B'}(6, -1). \tag{78}$$

Combining this with:

$$\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_{B'}, \quad R^1\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_{B'}(-f') \tag{79}$$

gives us formulas for the direct images of $\wedge^2 V_2$:

$$\pi_* \wedge^2 V_2 \simeq \mathcal{O}_{B'}(6, -1), \quad R^1\pi_* \wedge^2 V_2 \simeq \mathcal{O}_{B'}(6, -2). \tag{80}$$

We then push on to $\mathbb{P}^1$ as in (70), and since $R^1\beta'_* = 0$ for $k = 6$, we find:

$$\begin{aligned}
(\beta' \circ \pi)_* \wedge^2 V_2 &= \beta'_*(\pi_* \wedge^2 V_2) = \beta'_*\mathcal{O}_{B'}(6,-1) = S_x^6 \otimes \mathcal{O}_{\mathbb{P}^1_t}(-1) \oplus yS_x^4 \otimes \mathcal{O}_{\mathbb{P}^1_t}(-2) \\
R^1(\beta' \circ \pi)_* \wedge^2 V_2 &= \beta'_*(R^1\pi_* \wedge^2 V_2) = \beta'_*\mathcal{O}_{B'}(6,-2) = S_x^6 \otimes \mathcal{O}_{\mathbb{P}^1_t}(-2) \oplus yS_x^4 \otimes \mathcal{O}_{\mathbb{P}^1_t}(-3) \\
R^2(\beta' \circ \pi)_* \wedge^2 V_2 &= 0.
\end{aligned} \tag{81}$$

Since none of these sheaves have any global sections, we find the cohomology on $\tilde{X}$ by taking H^1 of the images on $\mathbb{P}^1_t$:

$$\begin{array}{ll}
H^0(\tilde{X}, \wedge^2 V_2) = 0, & h^0(\tilde{X}, \wedge^2 V_2) = 0, \\
H^1(\tilde{X}, \wedge^2 V_2) = yS_x^4, & h^1(\tilde{X}, \wedge^2 V_2) = 5, \\
H^2(\tilde{X}, \wedge^2 V_2) = S_x^6 \oplus yS_x^4 \otimes (S_t^1)^*, & h^2(\tilde{X}, \wedge^2 V_2) = 7 + 2 \times 5 = 17, \\
H^3(\tilde{X}, \wedge^2 V_2) = 0, & h^3(\tilde{X}, \wedge^2 V_2) = 0.
\end{array} \tag{82}$$

• $\boxed{\wedge^2 V_2^*}$ The cohomology of $\wedge^2 V_2^*$ can be obtained from that of $\wedge^2 V_2$ by Serre duality. Equivalently, we can apply the above procedure to $\wedge^2 V_2^* = \pi^*\mathcal{O}_{B'}(-6,1)$, noting that for $k = -6$ all the β'_* terms in (70) vanish:

$$\pi_* \wedge^2 V_2^* = \mathcal{O}_{B'}(-6,1), \quad R^1\pi_* \wedge^2 V_2^* = \mathcal{O}_{B'}(-6,0). \tag{83}$$

$$\begin{aligned}
(\beta' \circ \pi)_* \wedge^2 V_2^* &= 0, \\
R^1(\beta' \circ \pi)_* \wedge^2 V_2^* &= R^1\beta'_*(\pi_* \wedge^2 V_2^*) = R^1\beta'_*\mathcal{O}_{B'}(-6,1) \\
&= S_x^{4*} \otimes \mathcal{O}_{\mathbb{P}^1_t}(1) \oplus yS_x^{6*} \otimes \mathcal{O}_{\mathbb{P}^1_t}, \\
R^2(\beta' \circ \pi)_* \wedge^2 V_2^* &= R^1\beta'_*(R^1\pi_* \wedge^2 V_2^*) = R^1\beta'_*\mathcal{O}_{B'}(-6,0) \\
&= S_x^{4*} \otimes \mathcal{O}_{\mathbb{P}^1_t} \oplus yS_x^{6*} \otimes \mathcal{O}_{\mathbb{P}^1_t}(-1),
\end{aligned} \tag{84}$$

$$\begin{array}{rcll}
H^0(\tilde{X}, \wedge^2 V_2^*) &=& 0, & h^0(\tilde{X}, \wedge^2 V_2^*) = 0, \\
H^1(\tilde{X}, \wedge^2 V_2^*) &=& H^0(\mathbb{P}^1_t, S_x^{4*} \otimes \mathcal{O}_{\mathbb{P}^1_t}(1) \oplus yS_x^{6*} \otimes \mathcal{O}_{\mathbb{P}^1_t}) & \\
&=& S_x^{4*} \otimes S_t^1 \oplus yS_x^{6*}, & h^1(\tilde{X}, \wedge^2 V_2^*) = 5 \times 2 + 7 = 17, \\
H^2(\tilde{X}, \wedge^2 V_2^*) &=& H^0(\mathbb{P}^1_t, S_x^{4*} \otimes \mathcal{O}_{\mathbb{P}^1_t} \oplus yS_x^{6*} \otimes \mathcal{O}_{\mathbb{P}^1_t}(-1)) & \\
&=& S_x^{4*}, & h^2(\tilde{X}, \wedge^2 V_2^*) = 5, \\
H^3(\tilde{X}, \wedge^2 V_2^*) &=& 0, & h^3(\tilde{X}, \wedge^2 V_2^*) = 0.
\end{array} \tag{85}$$

• $\boxed{V_2 \otimes V_3^*}$ We recall that $C_2 = \overline{C}_2 \cup f_\infty$, and W_2 is related to $\overline{W}_2$ by sequence (51). If we tensor (51) by W_3^* and push to $\mathbb{P}^1$ with β_*, we find

$$0 \to \beta_*(i_{f_\infty *}G^* \otimes W_3^*) \to \mathcal{F} \to \overline{\mathcal{F}} \to 0, \tag{86}$$

where

$$\mathcal{F} := R^1\beta_*(W_2 \otimes W_3^*), \quad \overline{\mathcal{F}} := R^1\beta_*(\overline{W}_2 \otimes W_3^*), \tag{87}$$

and the last term in (86) is 0 because G^* has degree $+1$ on f_∞. All the sheaves in (86) have finite support:

- $\overline{\mathcal{F}}$ is supported on $\beta(\overline{C}_2 \cap C_3)$. If we choose things generically, $\overline{C}_2 \cap C_3$ will consist of 12 points p_j in B', the image $\beta(\overline{C}_2 \cap C_3)$ will consist of 12 distinct points $\hat{p}_j := \beta(p_j) \in \mathbb{P}^1$, $j = 1, \ldots, 12$, and $\overline{\mathcal{F}}$ will decompose as the sum of 12 rank 1 skyscraper sheaves $\mathcal{F}_j$ near each $\hat{p}_j$: $\overline{\mathcal{F}} = \bigoplus_{j=1}^{12} \mathcal{F}_j$.

- $\beta_*(i_{f_\infty *}G^* \otimes W_3^*)$ is supported at $\infty \in \mathbb{P}^1_t$, and has rank 3 there. It can therefore be decomposed (non-canonically) as $\bigoplus_{j=13}^{15} \mathcal{F}_j$, with each $\mathcal{F}_j$ a rank 1 skyscraper sheaf supported at ∞. For $j = 13, 14, 15$ we use $\hat{p}_j$ as another notation for the point $\infty \in \mathbb{P}^1_t$, the support of $\mathcal{F}_j$.

- The sequence (86) splits, so $\mathcal{F} = \bigoplus_{j=1}^{15} \mathcal{F}_j$.

We can now combine this with formula (70) applied to $L_2 \otimes L_3^* = \mathcal{O}_{B'}(5, 0)$, to compute $H^1(\tilde{X}, V_2 \otimes V_3^*)$:

$$\begin{aligned} H^1(\tilde{X}, V_2 \otimes V_3^*) &= H^0(\mathbb{P}^1_t, R^1\beta_*(W_2 \otimes W_3^*) \otimes \beta'_*(L_2 \otimes L_3^*)) \\ &= H^0(\mathbb{P}^1_t, \mathcal{F} \otimes [S_x^5 \otimes \mathcal{O}_{\mathbb{P}^1_t} \oplus y S_x^3 \otimes \mathcal{O}_{\mathbb{P}^1_t}(-1)]) \\ &= \bigoplus_{j=1}^{15} H^0(\mathbb{P}^1_t, \mathcal{F}_j) \otimes [S_x^5 \oplus y S_x^3 \otimes \{\hat{p}_j \mathbb{C}\}]. \end{aligned} \tag{88}$$

Here, we use the notation $\{\hat{p}_j \mathbb{C}\} \subset S_t^{1*}$ for the line inside the 2-dimensional plane S_t^{1*} consisting of all points proportional to $\hat{p}_j \in \mathbb{P}^1_t = \mathbb{P}(S_t^{1*})$. This line is the fiber at $\hat{p}_j$ of the line bundle $\mathcal{O}_{\mathbb{P}^1_t}(-1)$. In particular, the dimension is

$$h^1(\tilde{X}, V_2 \otimes V_3^*) = 150 = 15 \times [6 + 4]. \tag{89}$$

• $\boxed{V_2^* \otimes V_3^*}$ From the Chern character formula (52) we know that $W_2^* \simeq W_2 \otimes \mathcal{O}_{B'}(f)$, and therefore

$$R^1\beta_*(W_2^* \otimes W_3^*) \simeq R^1\beta_*(\beta^*\mathcal{O}_{\mathbb{P}^1_t}(1) \otimes W_2 \otimes W_3^*) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1_t}(1). \tag{90}$$

In analogy with (88) we therefore get

$$H^2(\tilde{X}, V_2^* \otimes V_3^*) \quad = \quad H^0(\mathbb{P}^1_t, R^1\beta_*(W_2^* \otimes W_3^*) \otimes R^1\beta'_*(L_2^* \otimes L_3^*))$$

$$
\begin{aligned}
&= H^0(\mathbb{P}^1_t, \mathcal{F}\otimes[yS_x^{1*}]) \\
&= \bigoplus_{j=1}^{15} H^0(\mathbb{P}^1_t, \mathcal{F}_j)\otimes yS_x^{1*},
\end{aligned} \tag{91}
$$

and the dimension is

$$
h^2(\tilde{X}, V_2^*\otimes V_3^*) = 30 = 15\times 2. \tag{92}
$$

• $\boxed{\wedge^2\tilde{V}}$ We note that the short exact sequence (42) which defines $\tilde{V}$ implies the exact sequence

$$
0\to\wedge^2 V_2\to\wedge^2\tilde{V}\to Q\to 0\ , \tag{93}
$$

where Q is defined by the quotient of the map $\wedge^2 V_2\to\wedge^2\tilde{V}$. However, the natural map $\wedge^2\tilde{V}\to\wedge^2 V_3$ factors through Q with the kernel $V_2\otimes V_3$. A simple consistency check for this statement is by dimension counting. Recall that V_2, V_3 and $\tilde{V}$ have rank 2, 3 and 5 respectively. Then, Q has dimension $\frac{5\cdot 4}{2}-\frac{2\cdot 1}{2}=9$ from (93), $\wedge^2 V_3$ has dimension $\frac{3\cdot 2}{2}=3$, so the kernel should have dimension $9-3=6$. This is indeed the dimension of $V_2\otimes V_3$, which is $2\cdot 3=6$. In summary, we have an intertwined pair of short exact sequences as follows.

$$
\begin{array}{ccccccccc}
 & & & & & 0 & & & \\
 & & & & & \uparrow & & & \\
 & & & & & \wedge^2 V_3 & & & \\
 & & & & & \uparrow & & & \\
0 & \to & \wedge^2 V_2 & \to & \wedge^2\tilde{V} & \to & Q & \to & 0\ . \\
 & & & & & & \uparrow & & \\
 & & & & & & V_2\otimes V_3 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array} \tag{94}
$$

This then induces the following two long exact sequences in cohomology,

$$
\begin{array}{ccccccccc}
0 & \to & H^0(\tilde{X},\wedge^2 V_2) & \to & H^0(\tilde{X},\wedge^2\tilde{V}) & \to & H^0(\tilde{X},Q) & \to & \\
 & \to & H^1(\tilde{X},\wedge^2 V_2) & \to & \boxed{H^1(\tilde{X},\wedge^2\tilde{V})} & \to & H^1(\tilde{X},Q) & \to & \\
 & \to & H^2(\tilde{X},\wedge^2 V_2) & \to & H^2(\tilde{X},\wedge^2\tilde{V}) & \to & H^2(\tilde{X},Q) & \to & \\
 & \to & H^3(\tilde{X},\wedge^2 V_2) & \to & H^3(\tilde{X},\wedge^2\tilde{V}) & \to & H^3(\tilde{X},Q) & \to & 0\ ,
\end{array} \tag{95}
$$

and

$$
\begin{array}{ccccccccc}
0 & \to & H^0(\tilde{X},V_2\otimes V_3) & \to & H^0(\tilde{X},Q) & \to & H^0(\tilde{X},\wedge^2 V_3) & \to & \\
 & \to & H^1(\tilde{X},V_2\otimes V_3) & \to & H^1(\tilde{X},Q) & \to & H^1(\tilde{X},\wedge^2 V_3) & \to & \\
 & \to & H^2(\tilde{X},V_2\otimes V_3) & \to & H^2(\tilde{X},Q) & \to & H^2(\tilde{X},\wedge^2 V_3) & \to & \\
 & \to & H^3(\tilde{X},V_2\otimes V_3) & \to & H^3(\tilde{X},Q) & \to & H^3(\tilde{X},\wedge^2 V_3) & \to & 0\ .
\end{array} \tag{96}
$$

We have boxed $H^1(\tilde{X}, \wedge^2\tilde{V})$ since it is the term we wish to compute.

First consider the second sequence (96). By the same arguments as (73), we have that

$$H^0(\tilde{X}, V_2 \otimes V_3) = H^3(\tilde{X}, V_2 \otimes V_3) = H^0(\tilde{X}, \wedge^2 V_3) = H^3(\tilde{X}, \wedge^2 V_3) = 0 \ . \tag{97}$$

It then follows from (96) that

$$H^0(\tilde{X}, Q) = H^3(\tilde{X}, Q) = 0. \tag{98}$$

Furthermore, using the Leray spectral sequence and the fact that $\pi_* \wedge^2 V_3 = 0$ implies

$$H^1(\tilde{X}, \wedge^2 V_3) \simeq H^0(B', R^1\pi_* \wedge^2 V_3). \tag{99}$$

Now,

$$R^1\pi_* \wedge^2 V_3 = \beta'^*(R^1\beta_* \wedge^2 W_3) \otimes L_3^{\otimes 2}. \tag{100}$$

Therefore, pushing (100) down to $\mathbb{P}^1$, (99) becomes

$$H^0(B', R^1\pi_* \wedge^2 V_3) = H^0(\mathbb{P}^1, (R^1\beta_* \wedge^2 W_3) \otimes \beta'_* L_3^{\otimes 2}). \tag{101}$$

Using (53), we see that $L_3^{\otimes 2}$ has negative degree along a generic fiber. Therefore, assuming that the support of $R^1\beta_* \wedge^2 W_3$ is on irreducible fibers, $\beta'_* L_3^{\otimes 2}$ vanishes and

$$H^1(\tilde{X}, \wedge^2 V_3) = 0 \ . \tag{102}$$

Substituting (97) and (102) into (96) implies

$$H^1(\tilde{X}, Q) \simeq H^1(\tilde{X}, V_2 \otimes V_3) \ , \tag{103}$$

and that $H^2(\tilde{X}, Q)$ fits into the short exact sequence

$$0 \to H^2(\tilde{X}, V_2 \otimes V_3) \to H^2(\tilde{X}, Q) \to H^2(\tilde{X}, \wedge^2 V_3) \to 0. \tag{104}$$

Having established these results, let us now consider the first sequence (95). Substituting (98) into (95) gives

$$H^0(\tilde{X}, \wedge^2\tilde{V}) \simeq H^0(\tilde{X}, \wedge^2 V_2) \ , \tag{105}$$

and

$$0 \to H^1(\tilde{X}, \wedge^2 V_2) \to \boxed{H^1(\tilde{X}, \wedge^2\tilde{V})} \to H^1(\tilde{X}, Q) \to H^2(\tilde{X}, \wedge^2 V_2) \to \dots \tag{106}$$

Putting (103) into (106) then leads to the exact sequence

$$0 \to H^1(\tilde{X}, \wedge^2 V_2) \to \boxed{H^1(\tilde{X}, \wedge^2\tilde{V})} \to H^1(\tilde{X}, V_2 \otimes V_3) \xrightarrow{M^T} H^2(\tilde{X}, \wedge^2 V_2) \to \dots \tag{107}$$

with which we will determine the desired boxed term. In (107), we have explicitly labeled a map M^T, namely the coboundary map

$$M^T : H^1(\tilde{X}, V_2 \otimes V_3) \to H^2(\tilde{X}, \wedge^2 V_2) \ . \tag{108}$$

It is given by cup product with

$$[\tilde{V}] \in H^1(\tilde{X}, V_3^* \otimes V_2) = \mathrm{Ext}^1_{\tilde{X}}(V_3, V_2) \ , \tag{109}$$

the extension class of $\tilde{V}$, via the pairing

$$\begin{array}{rccccc} \mathcal{M}^T : & H^1(\tilde{X}, V_2 \otimes V_3) & \times & H^1(\tilde{X}, V_3^* \otimes V_2) & \to & H^2(\tilde{X}, \wedge^2 V_2) \\ & A & \times & B & \to & C \ . \end{array} \tag{110}$$

This can be dualized to

$$\begin{array}{rccccc} \mathcal{M} : & H^1(\tilde{X}, \wedge^2 V_2^*) & \times & H^1(\tilde{X}, V_3^* \otimes V_2) & \to & H^2(\tilde{X}, V_2^* \otimes V_3^*) \\ & C^* & \times & B & \to & A^* \end{array} \ . \tag{111}$$

In formulas (85), (88) and (91) we have expressed the three cohomology groups in (111) as H^0 on $\mathbb{P}^1_t$ of appropriate sheaves. The naturality of our construction implies that the multiplication map $\mathcal{M}$ on cohomologies is itself induced from the natural multiplication map of the underlying sheaves on $\mathbb{P}^1_t$, namely:

$$\left(S_x^{4*} \otimes \mathcal{O}_{\mathbb{P}^1_t}(1) \oplus y S_x^{6*} \otimes \mathcal{O}_{\mathbb{P}^1_t}\right) \otimes \left(\mathcal{F} \otimes [S_x^5 \otimes \mathcal{O}_{\mathbb{P}^1_t} \oplus y S_x^3 \otimes \mathcal{O}_{\mathbb{P}^1_t}(-1)]\right) \to \mathcal{F} \otimes y S_x^{1*}. \tag{112}$$

By taking global sections, we find that $\mathcal{M}$ is the product:

$$\mathcal{M} : \left(S_x^{4*} \otimes S_t^1 \oplus y S_x^{6*}\right) \otimes \left(\bigoplus_{j=1}^{15} H^0(\mathbb{P}^1_t, \mathcal{F}_j) \otimes [S_x^5 \oplus y S_x^3 \otimes \{\hat{p}_j \mathbb{C}\}] \right) \to \bigoplus_{j=1}^{15} H^0(\mathbb{P}^1_t, \mathcal{F}_j) \otimes y S_x^{1*}. \tag{113}$$

In particular, our $\mathcal{M}$ breaks into blocks. The three spaces involved in $\mathcal{M}$ have dimensions 17, 150 and 30 respectively. This breaks into 15 blocks ($j = 1, \ldots, 15$), each sending a 17×10 dimensional space to a 2-dimensional space. Each block breaks further into a $10 \times 4 \to 2$ sub-block and a $7 \times 6 \to 2$ sub-block, corresponding to the products

$$(S_x^{4*} \otimes S_t^1) \otimes (S_x^3 \otimes \{\hat{p}_j \mathbb{C}\}) \to S_x^{1*} \tag{114}$$

and

$$(S_x^{6*}) \otimes (S_x^5) \to S_x^{1*}, \tag{115}$$

respectively. (We have suppressed a $yH^0(\mathcal{F}_j)$ factor on each side). The transpose $M : C^* \to A^*$ of our map M^T is obtained from (113) by evaluating at the extension class $[\tilde{V}] \in B$. We can express this $[\tilde{V}]$ in terms of its coefficients $a_{i,j}$, $i = 0, \ldots, 5$, $j = 1, \ldots, 15$ and $b_{k,j}$, $k = 0, \ldots, 3$, $j = 1, \ldots, 15$, in the S_x^5 and S_x^3 factors respectively. Now the map $S_x^{6*} \to S_x^{1*}$ given by the $a_{i,j}$ is represented by the 2×6 matrix

$$M_{I,j} = \begin{pmatrix} a_{0,j} & \ldots & a_{5,j} & 0 \\ 0 & a_{0,j} & \ldots & a_{5,j} \end{pmatrix}, \tag{116}$$

while the map $S_x^{4*} \otimes S_t^1 \to S_x^{1*}$ given by the $b_{k,j}$ is represented by the 2×10 matrix

$$M_{II,j} = \left(\begin{array}{cccc|cccc} b_{0,j}t_0(\hat{p}_j) & \ldots & b_{3,j}t_0(\hat{p}_j) & 0 & b_{0,j}t_1(\hat{p}_j) & \ldots & b_{3,j}t_1(\hat{p}_j) & 0 \\ 0 & b_{0,j}t_0(\hat{p}_j) & \ldots & b_{3,j}t_0(\hat{p}_j) & 0 & b_{0,j}t_1(\hat{p}_j) & \ldots & b_{3,j}t_1(\hat{p}_j) \end{array} \right). \tag{117}$$

So the full 30×17 matrix M is then

$$M = \begin{pmatrix} M_{I,1} & M_{II,1} \\ \vdots & \vdots \\ M_{I,15} & M_{II,15} \end{pmatrix}. \tag{118}$$

For a generic choice of the $a_{i,j}$ and $b_{k,j}$, the rank of M is 17 and M is surjective. It is easy to see that this remains true also for generic τ-invariant extension $[\tilde{V}]$. Plugging this, along with formulas (85) and (92), into (107), we find:

$$h^1(\tilde{X}, \wedge^2\tilde{V}) = 5 + 30 - 17 = 18. \tag{119}$$

Using Serre duality on $\tilde{X}$ and the fact that $\operatorname{ind}(\tilde{V}) = \operatorname{ind}(\wedge^2\tilde{V}) = 6$ [36], it is now straightforward to determine the remaining cohomology groups of $\tilde{V}$, $\tilde{V}^*$, $\wedge^2\tilde{V}$ and $\wedge^2\tilde{V}^*$.

5 The $\mathbb{Z}_2$ Action

In subsection 3.3 we described the involutions τ_B, $\tau_{B'}$, τ acting compatibly on B, B' and $\tilde{X}$. The action of $\tau_{B'}$ on line bundles on B' is specified in Table 3. In particular, the line bundles $\mathcal{O}_{B'}(0,1)$ and $\mathcal{O}_{B'}(1,0)$ are τ-invariant. It follows that there are induced involutions $\tau_{\mathbb{P}^1_t}$, $\tau_{\mathbb{P}^1_x}$ that commute with the corresponding maps $\beta' : B' \to \mathbb{P}^1_t$, $\chi : B' \to \mathbb{P}^1_x$. We have already encountered the involution $\tau_{\mathbb{P}^1_t}$ in subsection 3.2, where we denoted it simply $\tau_{\mathbb{P}^1}$. It sends $t_0 \mapsto t_0, \quad t_1 \mapsto -t_1$. We claim that $\tau_{\mathbb{P}^1_x}$ is also a non-trivial involution, so with an appropriate choice of the coordinates x_0, x_1 on $\mathbb{P}^1_x$ (note that we never fixed these coordinates up till now!) it acts as $x_0 \mapsto x_0, \quad x_1 \mapsto -x_1$. For this, we must determine the action of τ on the $\mathbb{P}^1$ family of rational curves r'. For a general, non-singular member of this family, all we learn from Table 3 is that it goes to another such. But the table also tells us the image under $\tau_{B'}$ of each of the line bundles $\mathcal{O}_{B'}(K_i^d)$, as K_i^d runs over the 16 components of the 8 reducible curves in the system $|r'|$, specified in (58). Each of these has the property that K_i^d is the only effective curve in its class: $h^0(B', K_i^d) = 1$. So we can deduce from Table 3 not only the cohomology class of the image, but the actual physical image:

$$K_2^d \leftrightarrow K_3^d, \quad K_1^d \leftrightarrow K_9^d, \quad K_4^d \leftrightarrow K_7^{3-d}, \quad K_6^d \leftrightarrow K_8^{3-d}. \tag{120}$$

At any rate, this clearly demonstrates that $\tau_{\mathbb{P}^1_x}$ is not the identity, as claimed.

Via the map Δ, our surface B' is a double cover of $\mathsf{Q} = \mathbb{P}^1_t \times \mathbb{P}^1_x$. Its equation can be written explicitly as

$$y^2 = F_{4,2}(x,t), \tag{121}$$

with $F_{4,2}(x,t)$ a bihomogeneous polynomial, of degree 4 in x_0, x_1 and of degree 2 in t_0, t_1. By (64), y is a section of $\mathcal{O}_{B'}(2,1)$ which vanishes on the ramification locus Ram_Δ. Since Ram_Δ goes to itself under $\tau_{B'}$, y must go to a multiple of itself. Since $\tau_{B'}$ is an involution, this multiple is ± 1, so in particular $F_{4,2}$ must be invariant (rather than anti-invariant). From (64), it follows that either $\tau_{B'} y = y$ or $\iota\tau_{B'} y = y$. Both involutions $\tau_{B'}$, $\iota\tau_{B'}$ have the same properties. So by relabelling $\iota\tau_{B'}$ as $\tau_{B'}$ if necessary, we may as well assume that the action of $\tau_{B'}$ is given explicitly by:

$$t_0 \mapsto t_0, \quad t_1 \mapsto -t_1, \quad x_0 \mapsto x_0, \quad x_1 \mapsto -x_1, \quad y \mapsto y. \tag{122}$$

In subsection 3.4 we chose compatible actions of τ on V_2, V_3 and $\tilde{V}$. It turns out that the particle spectrum on X is independent of these choices and is precisely half the spectrum on $\tilde{X}$ which we computed above. We compute it as follows.

- $\boxed{H^1(\tilde{X}, \tilde{V})}$ We have identified $H^1(\tilde{X}, \tilde{V})$ with $H^0(f_\infty, G^*) \otimes H^0(f_0', L_2)$ in (75), (77). We plug $k = 3$, $l = 0$ into formula (66), and restrict the double cover $\Delta : B' \to \mathsf{Q}$ to $\chi : f_0' \to \mathbb{P}^1_x$, finding:

$$\chi_* \mathcal{O}_{f_0'}(3r') = \mathcal{O}_{\mathbb{P}^1_x}(3) \oplus y\mathcal{O}_{\mathbb{P}^1_x}(1). \tag{123}$$

We get a natural identification of $H^0(f_0', L_2) = H^0(f_0', 3r')$ with $S_x^3 \oplus yS_x^1$. From (122) we see that the τ action on this 6-dimensional space has a 3-dimensional invariant subspace and 3-dimensional anti-invariant subspace. There is also a τ-action on the 1-dimensional $H^0(f_\infty, G^*)$, which must be either invariant or anti-invariant. Either way, we find:

$$h^1(\tilde{X}, \tilde{V})_+ = 3, \quad h^1(\tilde{X}, \tilde{V})_- = 3. \tag{124}$$

- $\boxed{H^1(\tilde{X}, \wedge^2\tilde{V})}$ From the identification of $H^1(\tilde{X}, \wedge^2 V_2)$ with yS_x^4 in (82), we see that

$$h^1(\tilde{X}, \wedge^2 V_2)_+ = 3, \quad h^1(\tilde{X}, \wedge^2 V_2)_- = 2, \tag{125}$$

while the identification of $H^2(\tilde{X}, \wedge^2 V_2)$ with $S_x^6 \oplus yS_x^4 \otimes (S_t^1)^*$ gives

$$h^2(\tilde{X}, \wedge^2 V_2)_+ = 4 + 5 = 9, \quad h^2(\tilde{X}, \wedge^2 V_2)_- = 3 + 5 = 8. \tag{126}$$

On the other hand, we saw in (91) that $H^1(\tilde{X}, V_2 \otimes V_3)$ is dual to $\bigoplus_{j=1}^{15} H^0(\mathbb{P}_t^1, \mathcal{F}_j) \otimes (yS_x^{1*})$. Again, the action of τ on the 2-dimensional space yS_x^{1*} has 1-dimensional invariants and 1-dimensional anti-invariants, so regardless of its action on the 15 1-dimensional spaces $H^0(\mathbb{P}_t^1, \mathcal{F}_j)$, we get:

$$h^1(\tilde{X}, V_2 \otimes V_3)_+ = 15, \quad h^1(\tilde{X}, V_2 \otimes V_3)_- = 15. \tag{127}$$

Combining the last three formulae with (107) and recalling that M^T is τ-equivariant (since it is cup product with the class $[\tilde{V}]$, which was taken in subsection 3.4 to be τ-invariant), we see that for those generic choices to which (119) applies we have:

$$h^1(\tilde{X}, \wedge^2\tilde{V})_+ = 3 + 15 - 9 = 9, \quad h^1(\tilde{X}, \wedge^2\tilde{V})_- = 2 + 15 - 8 = 9. \tag{128}$$

- $\boxed{H^1(\tilde{X}, \tilde{V}^*) \textbf{ and } H^1(\tilde{X}, \wedge^2\tilde{V}^*)}$ The spectrum also requires the terms $H^1(\tilde{X}, \tilde{V}^*)$ and $H^1(\tilde{X}, \wedge^2\tilde{V}^*)$. These can be obtained from the three-family condition (C3) in (36), in conjunction with the index theorem (147), as well as Serre duality (142) presented in the Appendix. Together with the fact that $H^0(\tilde{X}, \tilde{V})$, $H^0(\tilde{X}, \tilde{V}^*) = H^3(\tilde{X}, \tilde{V})^*$, $H^0(\tilde{X}, \wedge^2\tilde{V})$, and $H^0(\tilde{X}, \wedge^2\tilde{V}^*) = H^3(\tilde{X}, \wedge^2\tilde{V})^*$ all vanish, we have that

$$-h^1(\tilde{X}, U_i(\tilde{V})) + h^1(\tilde{X}, U_i(\tilde{V}^*)) = 6, \qquad U_i(\tilde{V}) = \tilde{V}, \ \wedge^2 \tilde{V} \ . \tag{129}$$

In fact, a $\mathbb{Z}_2$-graded version of the index theorem implies the stronger result that

$$-h^1(\tilde{X}, U_i(\tilde{V}))_{(\pm)} + h^1(\tilde{X}, U_i(\tilde{V}^*))_{(\pm)} = 3, \qquad U_i(\tilde{V}) = \tilde{V}, \ \wedge^2 \tilde{V} \ . \tag{130}$$

Alternatively, we can think of it as the index theorem applied to each of the τ-invariant and anti-invariant pieces of the cohomology.

Therefore, combining (130) with (124), we have that

$$h^1(\tilde{X},\tilde{V}^*)_+ = 6, \quad h^1(\tilde{X},\tilde{V}^*)_- = 6. \tag{131}$$

Similarly, combining (130) with (128), we have that

$$h^1(\tilde{X},\wedge^2\tilde{V})_+ = 12, \quad h^1(\tilde{X},\wedge^2\tilde{V})_- = 12. \tag{132}$$

Let us summarize the conclusions of the last two sections. It is convenient to introduce the following notation. Consider, for example, the cohomology group $H^1(\tilde{X},\tilde{V})$. We showed in Section 4 and Section 5 that $h^1(\tilde{X},\tilde{V}) = 6$ and $h^1(\tilde{X},\tilde{V})_{(+)} = h^1(\tilde{X},\tilde{V})_{(-)} = 3$ respectively. Henceforth, we will express both of these facts by writing

$$H^1(\tilde{X},\tilde{V}) = \mathbb{C}^3_{(+)} \oplus \mathbb{C}^3_{(-)}. \tag{133}$$

Using this notation, we encapsulate the results of Section 4 and Section 5 in Table 4.

U_i	$H^q(\tilde{X},U_i(\tilde{V}))$	R_i	$h^q(\tilde{X},U_i(\tilde{V}))$	A_j	$\mathbb{C}^r_{(+)} \oplus \mathbb{C}^s_{(-)}$
1	$H^0(\tilde{X},\mathcal{O}_{\tilde{X}})$	24	1	0	$\mathbb{C}^1_{(+)}$
10	$H^1(\tilde{X},\wedge^2\tilde{V})$	5	18	0	$\mathbb{C}^9_{(+)}$
				1	$\mathbb{C}^9_{(-)}$
$\overline{10}$	$H^1(\tilde{X},\wedge^2\tilde{V}^*)$	$\overline{5}$	24	0	$\mathbb{C}^{12}_{(+)}$
				1	$\mathbb{C}^{12}_{(-)}$
5	$H^1(\tilde{X},\tilde{V})$	$\overline{10}$	6	0	$\mathbb{C}^3_{(+)}$
				1	$\mathbb{C}^3_{(-)}$
$\overline{5}$	$H^1(\tilde{X},\tilde{V}^*)$	10	12	0	$\mathbb{C}^6_{(+)}$
				1	$\mathbb{C}^6_{(-)}$

Table 4: The dimensions and $\mathbb{Z}_2$ actions on the cohomology spaces $H^q(\tilde{X},U_i(\tilde{V}))$.

6 Low Energy Spectrum

We know from the discussion in Section 2, and specifically from equation (30), that the multiplicities of the representations B_{ij} of the low energy gauge group are determined by $(H^q(\tilde{X},U_i(\tilde{V}))\otimes A_j)^{\rho'(F)}$, the invariant part of $H^q(\tilde{X},U_i(\tilde{V}))\otimes A_j$ under the joint action of $\mathbb{Z}_2$ on $H^q(\tilde{X}, U_i(\tilde{V}))$ and A_j. By combining the results in Table 2 with the $\mathbb{Z}_2$ action on the cohomology groups listed in Table 4, we can now compute the complete low energy spectrum of our theory. The associated multiplets descend to $X = \tilde{X}/\mathbb{Z}_2$ to form the $(SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$ particle physics spectrum. The results are listed in Table 5. The representation $R_i = 1$, corresponding to the moduli $H^0(\tilde{X}, \mathrm{ad}\tilde{V})$, is not presented.

R_i	A_j	$(H^q(\tilde{X}, U_i(\tilde{V})) \otimes A_j)^{\rho'(F)}$	B_{ij}
24	0	$\mathbb{C}^1_{(+)}$	$(8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0$
5	0	$\mathbb{C}^9_{(+)}$	$(3,1)_{-2}$
	1	$\mathbb{C}^9_{(-)}$	$(1,2)_3$
$\overline{5}$	0	$\mathbb{C}^{12}_{(+)}$	$(\overline{3},1)_2$
	1	$\mathbb{C}^{12}_{(-)}$	$(1,2)_{-3}$
$\overline{10}$	0	$\mathbb{C}^3_{(+)}$	$(3,1)_4 \oplus (1,1)_{-6}$
	1	$\mathbb{C}^3_{(-)}$	$(\overline{3},2)_{-1}$
10	0	$\mathbb{C}^6_{(+)}$	$(\overline{3},1)_{-4} \oplus (1,1)_6$
	1	$\mathbb{C}^6_{(-)}$	$(3,2)_1$

Table 5: The particle spectrum of the low-energy $(SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$ theory. The A_j correspond to characters of the $\mathbb{Z}_2$ representations on R_i. The $U(1)$ charges listed are $w = 3Y$.

To begin with, the spectrum contains one copy of vector supermultiplets transforming under $(SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$ as

$$(8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0. \tag{134}$$

Contained in these multiplets are the gauge connections for $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ respectively. Furthermore, it contains three families of quarks and lepton superfields, each family transforming as

$$(3,2)_1, \quad (\overline{3},1)_{-4}, \quad (\overline{3},1)_2 \tag{135}$$

and

$$(1,2)_{-3}, \quad (1,1)_6 \tag{136}$$

respectively. Each of these multiplets is a chiral superfield, none of which has a conjugate partner. However, there are additional chiral superfields in the spectrum. It follows from

Table 5 that these occur as conjugate pairs of the $(SU(3)_C \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$ representations

$$(3,1)_{-2}, \quad (1,2)_3 \tag{137}$$

and

$$(3,1)_4 \oplus (1,1)_{-6}, \quad (\overline{3},2)_{-1}. \tag{138}$$

These multiplets represent extra matter in the spectrum, such as Higgs and other exotic fields.

Let us explain how the quark/lepton fermions and conjugate pairs arise. Consider, for example, the B_{ij} representations $(\overline{3},2)_{-1}$ and $(3,2)_1$, corresponding to the $\overline{10}$ and 10 representations respectively. From Table 5, we see that there are 3 copies of $(\overline{3},2)_{-1}$ and 6 copies of $(3,2)_1$. Note that $6-3=3$ copies of $(3,2)_1$ are unpaired, as a consequence of the index theorem. Each unpaired $(3,2)_1$ multiplet contributes to a single quark/lepton generation, as in (135). This leaves 3 conjugate pairs of $(\overline{3},2)_{-1}$ and $(3,2)_1$ superfields. Being non-chiral pairs, these do not contribute to a quark/lepton family but, rather, are additional supermultiplets as listed in (137) and (138).

It remains to enumerate the number of additional superfields. From Table 5, we see that the spectrum has

$$n_{(3,1)_{-2}} = 9, \quad n_{(1,2)_3} = 9 \tag{139}$$

and

$$n_{(3,1)_4 \oplus (1,1)_{-6}} = 3, \quad n_{(\overline{3},2)_{-1}} = 3 \tag{140}$$

copies of (137) and (138) respectively. The multiplicity $n_{(1,2)_3}$ corresponds to the number of Higgs doublet conjugate pairs in the low energy spectrum. The remaining multiplets in (137) and (138) are exotic.

We conclude that the low energy spectrum of the simple, representative model discussed in this paper includes the requisite three chiral families of quarks and leptons. Additionally, it naturally has Higgs doublet supermultiplet pairs. Unfortunately, the spectrum contains extra, exotic chiral supermultiplets which, potentially, are phenomenologically unacceptable. However, these conjugate pairs of exotic multiplets may couple to the moduli fields coming from $H^1(X, V \otimes V^*)$ to form mass terms. If the moduli can acquire a sufficiently high vacuum expectation value, then the exotics multiplets will decouple at low energy and be compatible with phenomenology. These couplings will be discussed elsewhere.

Armed with the technology developed in this paper, one can now compute the spectra of standard-like models based on arbitrary stable vector bundles on a wide range of elliptically fibered Calabi-Yau threefolds. These spectra can be constrained to always contain three families of quarks and leptons. We are presently searching for such vacua with a phenomenologically acceptable number of Higgs doublets and, hopefully, no exotic matter.

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A Some Useful Mathematical Facts

In this Appendix, we present some useful mathematical facts used throughout the paper [60, 61, 62]. The first is Serre duality, which implies that for a sheaf $\mathcal{F}$ on an n-fold X

$$H^q(X,\mathcal{F}) \simeq H^{n-q}(X,\mathcal{F}^* \otimes K_X)^*, \tag{141}$$

where K_X is the canonical bundle of X. For our Calabi-Yau threefold $\tilde{X}$ and sheaf $U_i(\tilde{V})$ on $\tilde{X}$, (141) simplifies to

$$H^q(\tilde{X},U_i(\tilde{V})) \simeq H^{3-q}(\tilde{X},U_i(\tilde{V})^*)^*, \tag{142}$$

where we have used the fact that $K_{\tilde{X}}$ on a Calabi-Yau manifold is trivial.

The second tool we use is the Atiyah-Singer index theorem, which implies that on our Calabi-Yau threefold $\tilde{X}$

$$\mathrm{ind}(U_i(\tilde{V}) = \sum_{q=0}^{3}(-1)^q h^q(\tilde{X},U_i(\tilde{V})) = \int_{\tilde{X}} \mathrm{ch}(U_i(\tilde{V})) \wedge \mathrm{td}(\tilde{X}) = \frac{1}{2}\int_{\tilde{X}} c_3(U_i(\tilde{V}))\ . \tag{143}$$

The three-generation condition means that on X, $\mathrm{ind}(V)$ is equal to three [1], which implies that on the cover $\tilde{X}$ [49, 50],

$$\mathrm{ind}(\tilde{V}) = |\mathbb{Z}_2| \times 3 = 6, \tag{144}$$

or,

$$c_3(\tilde{V}) = 12. \tag{145}$$

This is the origin of the condition (C3) in (36).

In this paper, we apply the index theorem in the two cases $U_i(\tilde{V}) = \tilde{V}$ and $\wedge^2\tilde{V}$. It was shown in Appendix A of [36] that for our $SU(5)$ bundle $\tilde{V}$

$$c_3(\wedge^2\tilde{V}) = c_3(\tilde{V}) = 12. \tag{146}$$

Therefore, in these cases, (143) simplifies to

$$\sum_{q=0}^{3}(-1)^q h^q(\tilde{X}, U_i(\tilde{V})) = 6, \qquad U_i(\tilde{V}) = \tilde{V}, \;\; \wedge^2\,\tilde{V}\ . \tag{147}$$

An important tool for computing cohomology groups of vector bundles or, more generally, coherent sheaves on fibered spaces is the Leray spectral sequence. Consider the map $\pi : \tilde{X} \to B'$ and a sheaf $\mathcal{F}$ on $\tilde{X}$. The Leray spectral sequence for the map π will relate the cohomologies of $\mathcal{F}$ on $\tilde{X}$ to the cohomologies of the higher direct image sheaves $R^i\pi_*\mathcal{F}$ on B'. For a general map, the sequence is rather complicated. However, in the case of π being an elliptic fibration, it will degenerate to a simpler long exact sequence.

To begin with, consider the definition of $R^0\pi_*\mathcal{F} = \pi_*\mathcal{F}$. It is a sheaf on B' given by

$$\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U)) = H^0(\pi^{-1}(U), \mathcal{F}|_{\pi^{-1}(U)}) \tag{148}$$

for any open set $U \subset B'$. The definition (148) generalizes to the higher image sheaves as

$$R^i\pi_*\mathcal{F}(U) = H^i(\pi^{-1}(U), \mathcal{F}|_{\pi^{-1}(U)})\ , \tag{149}$$

for sufficiently small U. It follows that for the map $\pi : \tilde{X} \to B'$

$$R^i\pi_*\mathcal{F}(U) = 0, \qquad i > \dim\pi^{-1}(U)\ . \tag{150}$$

In our case, the Leray spectral sequence degenerates to the long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^1(B', \pi_*\mathcal{F}) & \to & H^1(\tilde{X}, \mathcal{F}) & \to & H^0(B', R^1\pi_*\mathcal{F}) & \to & \\ & \to & H^2(B', \pi_*\mathcal{F}) & \to & H^2(\tilde{X}, \mathcal{F}) & \to & H^1(B', R^1\pi_*\mathcal{F}) & \to & 0\ . \end{array} \tag{151}$$

Note that $H^3(B', \pi_*\mathcal{F}) = 0$ since $\dim_{\mathbb{C}} B' = 2$. As promised, (151) relates the cohomology of $\mathcal{F}$ on $\tilde{X}$ to the cohomology of the higher image sheaves $R^i\pi_*\mathcal{F}$ on B'. Recall that B' is itself elliptically fibered. Therefore, one can write a Leray spectral sequence for the map $\beta' : B' \to \mathbb{P}^1$ in complete analogy to (151).

Another useful formula is Groethendieck-Riemann-Roch (GRR), which states that for any map $f : X \to B$ and any sheaf $\mathcal{S}$ on X, we have

$$\mathrm{td}(TB)\mathrm{ch}(\sum_{i=0}^{2}(-1)^i R^i f_*\mathcal{S}) = f_*(\mathrm{ch}(\mathcal{S})\mathrm{td}(TX))\ . \tag{152}$$

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