Lévy processes induced by Dirichlet (B-) splines: modelling multivariate asset price dynamics

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Abstract

We consider a new class of processes, called LG processes, defined as linear combinations of independent gamma processes. Their distributional and path-wise properties are explored by following their relation to polynomial and Dirichlet (B-) splines. In particular, it is shown that the density of an LG process can be expressed in terms of Dirichlet (B-) splines, introduced independently by Ignatov and Kaishev (1987, 1988, 1989a,b) and Karlin et al. (1986). We further show that the well known variance-gamma (VG) process, introduced by Madan and Seneta (1990), and the Bilateral Gamma (BG) process, recently considered by Küchler and Tappe (2008) are special cases of an LG process. Following this LG interpretation, we derive new (alternative) expressions for the VG and BG densities and consider their numerical properties. The LG process has two sets of parameters, the B-spline knots and their multiplicities, and offers further flexibility in controlling the shape of the Levy density, compared to the VG and the BG processes. Such flexibility is often desirable in practice, which makes LG processes interesting for financial and insurance applications.

Multivariate LG processes are also introduced and their relation to multivariate Dirichlet and simplex splines is established. Expressions for their joint density, the underlying LG-copula, the characteristic, moment and cumulant generating functions are given. A method for simulating LG sample paths is also proposed, based on the Dirichlet bridge sampling of

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Gamma processes, due to Kaishev and Dimitrova (2009). A method of moments for estimation of the LG parameters is also developed. Multivariate LG processes are shown to provide a competitive alternative in modelling dependence, compared to the various multivariate generalizations of the VG process, proposed in the literature. Application of multivariate LG processes in modelling the joint dynamics of multiple exchange rates is also considered.

Keywords: LG (Lévy) process; (multivariate) variance gamma process; bilateral gamma process; Dirichlet spline; B-spline; simplex spline; Dirichlet bridge sampling; cumulants; FX modelling;

1 Introduction

An important strand of literature on financial modelling in recent years is devoted to developing more realistic stochastic models incorporating appropriate Lévy processes as drivers of the price dynamics of financial assets. Examples of such processes are the Variance Gamma (VG) process introduced by Madan and Seneta (1990) (see also Madan et al. 1998) and the so called Bilateral Gamma (BG) process considered recently by Küchler and Tappe (2008). The three parameter VG process of Madan et al. (1998) is constructed by randomly changing the time in a Brownian motion with certain drift and volatility parameters, following a Gamma process with unit mean rate and certain variance rate parameter. The BG process is a generalization of the VG process and its increments have a four parameter Bilateral Gamma distribution, which represents two Gamma distributions, one for the positive and one for the negative half-lines, adjoined together at the origin. Both VG and BG processes are pure jump, infinite activity, finite-variation, Lévy processes, that inherit these properties from the Gamma processes underlying their construction. For an excellent account on properties of Gamma processes which play an important role throughout this paper, we refer to Yor (2007).

The exponential VG process has proved a successful alternative to Geometric Brownian motion in a number of applications, for example in option pricing (see Kaishev and Dimitrova 2009 and the references therein) and in credit risk modelling (see Schoutens and Cariboni 2009). The ability of the VG process to capture both upward and downward jumps as well as very small movements (jitters) in stock prices have been highlighted by Stein et al. (2007) who give an extensive list of further references on the VG model and its applications.

Many real life financial applications require modelling the joint dynamics of multiple, possibly dependent asset price processes. A typical example would be the necessity to model the joint movement of foreign currencies exchange rates. In such cases, developing models involving appropriate multivariate Lévy processes, capable of capturing different dependence patterns is of utmost importance. In order to meet such demands, attempts to extend the VG model to more than one dimension have been undertaken in several directions. In their seminal paper, Madan and Seneta (1990) propose a multivariate VG process, defined through a multivariate correlated Brownian motion, subordinated by a common Gamma process representing the common stochastic business clock. There are two sources of dependence in this model, one is the common Gamma clock and the second one is the correlation between the Brownian motions. We refer to this model as Common Clock Variance Gamma (CCVG) model (cf. Deelstra and Petkovic 2010). Luciano and Schoutens (2006) considered a special case of a CCVG model with zero correlation between the univariate Brownian motions. The level of dependence in this construction is controlled only through the common Gamma variance rate parameter which imposes some limitations on its flexibility (see the numerical illustration in section 4).

Further generalizations of this construction, due to Luciano and Semeraro (2007) and Semeraro (2008), allow for a decomposition of the time change in a common and idiosyncratic parts. Calibration to option pricing of CCVG with non-zero correlation is performed in Leoni and Schoutens (2008). For an overview of other constructions based on multivariate Lévy processes, in the context of option pricing, see Deelstra and Petkovic (2010).

The univariate BG process with its four parameters offers somewhat extended flexibility, compared to the univariate VG. However, to the best of our knowledge, no multivariate versions of the BG process have been considered in the literature.

In this paper we propose a new class of Lévy processes defined as linear combinations of independent Gamma processes. In what follows, it will be convenient to refer to such linear combinations as *LG processes.* It is directly verified (see section 2) that both the Variance Gamma process and the Bilateral Gamma process are special cases of an LG process represented as particular linear combinations of two Gamma processes.

Our aim in this paper is to introduce univariate and multivariate LG processes, explore their properties and illustrate how they can be applied in modelling the joint behavior of empirical asset price processes. As the VG and the BG, LG processes also preserve some of the nice features of the Gamma processes used for their construction. They are pure jump Lévy processes of finite variation which may jump infinitely many times on a finite time interval. We show that LG processes are intrinsically related to the so called Dirichlet splines and polynomial B-splines, and posses some of their interesting geometric properties. In particular, we give explicit expressions, in terms of multivariate Dirichlet (B-) splines, of the joint density of the LG distribution, generating multivariate LG processes.

Dirichlet splines, which have been independently introduced by Karlin et al. (1986) and by Ignatov and Kaishev (1987, 1988, 1989a,b) who call them *generalized B-splines*, are densities of linear transformations of Dirichlet random variables. When the shape parameters of the underlying Gamma processes are integer, the corresponding LG density is expressed in terms of multivariate simplex splines, introduced by de Boor (1976). We give also some new expressions, in terms of univariate Dirichlet (B-) splines, for the densities of the VG and BG distributions. The proposed approach allows for the uniform treatment of the wide class of LG processes in terms of multivariate Dirichlet (B-) splines for which methods of their efficient numerical evaluation exist (see section 3).

The structure of the paper is as follows. In section 2, we introduce univariate LG processes, note their relation to the Variance Gamma and Bilateral Gamma processes, explore their distributional properties and give the Lévy triplet and martingale conditions, which characterize them. In section 3, we introduce the multivariate version of an LG process, establish expressions in terms of multivariate Dirichlet (B-) splines for the joint density of its underlying joint LG distribution, give its underlying LG copula, its characteristic, moment and cumulant generating functions.

We also provide a method of moments, based on expressing them in terms of cumulants, for estimating the LG parameters. In Section 4, we illustrate how the multivariate LG processes are applied in modelling the dynamics of the joint movement of the exchange rates of a set of currencies. Section 5 provides conclusions and some further comments.

2 Linear combinations of Gamma (LG-) processes

Our aim here will be to consider a new class of stochastic processes, defined as linear combinations of independent Gamma processes and explore their distributional and path-wise properties.

For the purpose, denote by $G_i\left(t;\alpha_i,\lambda\right),\ i=0,\ldots,n$ a collection of $n+1$ independent *Gamma* processes, defined on a probability space $(\Omega,\mathfrak{F},\mathbb{P})$, with mean rate $\alpha_i/\lambda>0$ and variance rate $\alpha_i/\lambda^2>0$, where $\alpha_i>0$ and $\lambda>0$, $i=0,\ldots,n$. For a fixed t, $t>0$, the density of $G_i\left(t;\alpha_i,\lambda\right)$ is

$$f_{G_i}\left(x;\alpha_i,\lambda,t\right)=\frac{\lambda^{\alpha_i t}}{\Gamma\left(\alpha_i t\right)}x^{\alpha_i t-1}e^{-\lambda x},$$

where $x>0$. Let us recall that the Gamma process, $G_i\left(t;\alpha_i,\lambda\right)$, is a pure jump, finite variation process which jumps infinitely many times up to time t and has independent, gamma distributed increments. It plays a central role in contemporary financial modelling. For a detailed account on the properties of Gamma processes and their application in finance and insurance, we refer to Yor (2007), Fu (2007), Dufresne et al. (1991), Dickson and Waters (1993), Madan et al. (1998). We will use the gamma processes, $G_i\left(t;\alpha_i,\lambda\right)$, $i=0,\ldots,n$, as building blocks and define the process of interest in this paper, as follows.

Definition 2.1 *Given a set of real-valued parameters $\delta=\{\delta_0,\ldots,\delta_n\}$, define the process $LG(t;\delta,\alpha,\lambda,n)$ as a linear combination of the independent gamma processes, $G_i\left(t;\alpha_i,\lambda\right)$, $i=0,\ldots,n$, i.e.,*

$$LG(t;\delta,\alpha,\lambda,n)=\delta_0G_0\left(t;\alpha_0,\lambda\right)+\ldots+\delta_nG_n\left(t;\alpha_n,\lambda\right), \tag{2.1}$$

where $\alpha = \{\alpha_0, \ldots, \alpha_n\}$. For the sake of brevity we call such linear combinations, LG processes.

In Definition 2.1 we allow δ_i to coalesce. However, for notational convenience and without loss of generality in the sequel we shall assume that δ_i, $i = 0, \ldots, n$, are pairwise distinct.

In what follows, we shall sometimes abbreviate $LG(t; \delta, \alpha, \lambda, n)$ to $LG(t)$ and the two notations will be used interchangeably.

Let us note that the three parameter Variance Gamma process, introduced by Madan et al. (1998), is a special case of an LG process. To see this recall that the VG process, $VG(t; \theta, \sigma, \nu)$ is defined as

$$VG(t; \theta, \sigma, \nu) = B\left(G\left(t; \frac{1}{\nu}, \frac{1}{\nu}\right); \theta, \sigma\right),$$

where $B(t; \theta, \sigma)$ is a Brownian motion with drift $\theta \in \mathbb{R}$ and volatility $\sigma > 0$, and $G\left(t; \frac{1}{\nu}, \frac{1}{\nu}\right)$ is a Gamma process with mean rate 1 and variance rate $\nu > 0$. It is not difficult to see that the VG process admits the alternative, LG representation

$$VG(t; \theta, \sigma, \nu) = \delta_0 G_0\left(t; \alpha_0, 1\right) + \delta_1 G_1\left(t; \alpha_1, 1\right), \tag{2.2}$$

where $\delta_0 = -\frac{\sqrt{\theta^2 + 2\sigma^2/\nu} - \theta}{2}\nu$, $\delta_1 = \frac{\sqrt{\theta^2 + 2\sigma^2/\nu} + \theta}{2}\nu$; $\alpha_0 = \alpha_1 = \frac{1}{\nu}$, which is a special case of an $LG(t; \delta, \alpha, \lambda, n)$ process with $\lambda = 1$ and $n = 1$.

Equality (2.2) follows from the fact that the characteristic function of the VG process (see Madan et al. 1998), can be expressed as

$$\begin{aligned}\phi_{VG(t)}(u) = \left(\frac{1}{1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2}\right)^{\frac{1}{\nu}t} &= \left(\frac{1}{1 + iu\left|\delta_0\right|}\right)^{\alpha_0 t}\left(\frac{1}{1 - iu\delta_1}\right)^{\alpha_1 t} \\ = \mathbb{E}\left[e^{iu\{\delta_0 G_0(t;\alpha_0,1) + \delta_1 G_1(t;\alpha_1,1)\}}\right],\end{aligned}$$

where $|\delta_0|$ is the absolute value of δ_0. Furthermore, it is straightforward to see that a linear combination of say, p, VG processes is also a LG process.

It can be shown that the *Bilateral Gamma*(BG) process, recently considered by Küchler and Tappe (2008), is also a special case of an LG process. The BG processes are associated with the bilateral gamma distribution, $\Gamma\left(\alpha^+, \lambda^+, \alpha^-, \lambda^-\right)$, with parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^- > 0$, defined as the convolution

$$\Gamma\left(\alpha^+, \lambda^+, \alpha^-, \lambda^-\right) := \Gamma\left(\alpha^+, \lambda^+\right) * \Gamma\left(\alpha^-, -\lambda^-\right),$$

where $\Gamma(\alpha, \lambda)$ is a generalized Gamma distribution with parameters $\alpha > 0$, $\lambda \in \mathbb{R}\backslash\{0\}$. The density of $\Gamma(\alpha, \lambda)$ is given by

$$f_{BG}(x; \alpha, \lambda) = \frac{|\lambda|^\alpha}{\Gamma(\alpha)} |x|^{\alpha-1} e^{-|\lambda||x|} \left(\mathbf{1}_{\{\lambda>0\}}\mathbf{1}_{\{x>0\}} + \mathbf{1}_{\{\lambda<0\}}\mathbf{1}_{\{x<0\}}\right), \tag{2.3}$$

where $x \in \mathbb{R}$ and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. As can be seen from (2.3), when $\lambda > 0$, this is the well-known Gamma distribution, concentrating mass on $\mathbb{R}_+$, whereas, for $\lambda < 0$, the generalized Gamma distribution is simply a Gamma distribution on the negative half axis, $\mathbb{R}_-$. The corresponding bilateral gamma process, $BG\left(t; \alpha^+, \lambda^+, \alpha^-, \lambda^-\right)$ is a pure jump Lévy process, whose increments have bilateral gamma distribution and in particular, for fixed t, $t > 0$,

$$BG\left(t; \alpha^+, \lambda^+, \alpha^-, \lambda^-\right) \sim \Gamma\left(\alpha^+ t, \lambda^+, \alpha^- t, \lambda^-\right).$$

For further properties of the BG distribution and processes, and some applications in finance, we refer to Küchler and Tappe (2008).

It is directly verified that, the BG process is a four parameter generalization of the VG process and admits the following representation as an LG process

$$BG\left(t; \alpha^+, \lambda^+, \alpha^-, \lambda^-\right) = \delta_0 G_0\left(t; \alpha_0, 1\right) + \delta_1 G_1\left(t; \alpha_1, 1\right),$$

where $\delta_0 = -1/\lambda^-$; $\delta_1 = 1/\lambda^+$; $\alpha_0 = \alpha^-$, $\alpha_1 = \alpha^+$, $\lambda = 1$ and $n = 1$. As in the case of VG, linear combinations of BG processes are also LG processes.

2.1 Distributional properties

From Definition 2.1, for fixed t, say $t = 1$, it is directly seen that the characteristic function, $\phi_{LG(z)} = \mathbb{E}\left[e^{izLG(t)}\right]$, of a LG process is given by

$$\phi_{LG}(z) = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - i\delta_j z}\right)^{\alpha_j}, z \in \mathbb{R}.$$

The cumulant generating function, $\Psi(u) = \ln \mathbb{E}\left[e^{uLG(t)}\right]$, $u \in \mathbb{R}$ is

$$\Psi(u) = \sum_{j=0}^{n} \alpha_j \ln \frac{\lambda}{\lambda - \delta_j u},$$

where

$$\frac{\lambda}{\max_{j \in D_-}\{\delta_j\}} < u < \frac{\lambda}{\max_{j \in D_+}\{\delta_j\}},$$

$D_- = \{i \in I : \delta_i < 0\}, D_+ = \{i \in I : \delta_i \geq 0\}, I = \{1, \ldots, n\}$. The cumulants $\kappa_w = \Psi^{(w)}(0)$, where

$$\Psi^{(w)}(u) = (w-1)! \sum_{j=0}^{n} \alpha_j \delta_j^w (\lambda - \delta_j u)^{-w}, w = 1, 2, \ldots$$

are then obtained as

$$\kappa_w = (w-1)! \sum_{j=0}^{n} \frac{\alpha_j}{\lambda^w} \delta_j^w, w = 1, 2, \ldots \tag{2.4}$$

We can now use (2.4) and specify the mean, μ_{LG}, the variance, ν_{LG}, the Charliers skewness, χ_{LG}, and the kurtosis, τ_{LG}, of $LG(t)$ as

$$\mathbb{E}[LG(t)] = \mu_{LG} = \kappa_1 = \sum_{i \in D_+} \frac{\alpha_i \delta_i}{\lambda} - \sum_{i \in D_-} \frac{\alpha_i |\delta_i|}{\lambda},$$

$$\mathrm{Var}[LG(t)] = \nu_{LG} = \kappa_2 = \sum_{i \in D_+} \frac{\alpha_i \delta_i^2}{\lambda^2} + \sum_{i \in D_-} \frac{\alpha_i |\delta_i|^2}{\lambda^2}$$

$$\chi_{LG} = \kappa_3 / (\kappa_2)^{3/2} = \sum_{j=0}^{n} 2\alpha_j \delta_j^3 \lambda^{-3} / \left(\sum_{j=0}^{n} \alpha_j \delta_j^2 \lambda^{-2} \right)^{3/2}$$

$$\tau_{LG} = 3 + \kappa_4 / (\kappa_2)^2 = 3 + \sum_{j=0}^{n} 6\alpha_j \delta_j^4 \lambda^{-4} / \left(\sum_{j=0}^{n} \alpha_j \delta_j^2 \lambda^{-2} \right)^2 .$$

Let us now give an expression for the density of $LG(t)$. For the purpose, we will need some notation and background results. Denote by

$$S_n = \left\{ \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \text{for all } i, \sum_{i=1}^{n} x_i \leq 1 \right\},$$

the standard n-simplex and recall that the random vector $(\theta_0, \ldots, \theta_n)$, has Dirichlet distribution $\mathfrak{D}(\alpha_0, \ldots, \alpha_n)$ on S_n, with (real) parameters $\alpha_0 > 0$, $\ldots$, $\alpha_n > 0$, i.e., $(\theta_0, \ldots, \theta_n) \in \mathfrak{D}(\alpha_0, \ldots, \alpha_n)$, if $\theta_0 = 1 - \theta_1 - \ldots - \theta_n$ and the joint probability density of $\theta_1, \ldots, \theta_n$ is

$$f_{\theta_1,\ldots,\theta_n}(\boldsymbol{x}) = \frac{\Gamma(\alpha_0 + \ldots + \alpha_n)}{\prod_{i=0}^n \Gamma(\alpha_i)} \prod_{j=0}^n x_j^{\alpha_j - 1} \mathbf{1}_{\{\boldsymbol{x} \in S_n\}},$$

where $x_0 = 1 - x_1 - \ldots - x_n$. We will use the shorter notation $(\theta_0, \ldots, \theta_n) \in \mathfrak{D}(1)$ if $\alpha_j = 1$, $j = 0, \ldots, n$. We will now establish the following property of a LG process, which will be used in the sequel.

Lemma 2.2 *For a fixed t, $t > 0$, the process $LG(t; \delta, \alpha, \lambda, n)$, defined in (2.1), admits the representation*

$$LG(t; \delta, \alpha, \lambda, n) = B(t)\Gamma(t), \tag{2.5}$$

where $\Gamma(t) = \sum_{i=0}^n G_i(t; \alpha_i, \lambda)$, $B(t) = \delta_0\theta_0 + \ldots + \delta_n\theta_n$ and the random variables $\theta_0, \ldots, \theta_n$, have Dirichlet distribution $\mathfrak{D}(\alpha_0 t, \ldots, \alpha_n t)$ with (real) parameters $\alpha_0 t > 0, \ldots, \alpha_n t > 0$, i.e., $(\theta_0, \ldots, \theta_n) \in \mathfrak{D}(\alpha_0 t, \ldots, \alpha_n t)$ and $B(t)$ is independent of $\Gamma(t)$.

Proof: Representation (2.5) follows from the fact that, for fixed t, the r.v.s $\theta_0, \ldots, \theta_n$, coincide in distribution with the random variables $G_i(t; \alpha_i, \lambda)/\Gamma(t)$, $i = 0, \ldots, n$ (see e.g. Wilks 1962), and by the theorem of Sukhatme (1937), the latter are independent of $\Gamma(t)$ which yields the independence of $B(t)$ and $\Gamma(t)$. □

Lemma 2.2 is fundamental in the study of LG processes since it links their underlying LG distribution to the classical polynomial splines and in general to the so called, generalized B-splines (known also as Dirichlet splines). This link, as will be demonstrated, provides a different, spline-approximation insight into the distributional properties of LG processes. It is interesting, both from the theoretical and numerical point of view, since the theory of polynomial spline functions is well developed (see e.g. Schumaker 1981) and offers also numerically efficient recurrence formulas for the evaluation of (B-)splines (see de Boor 2001) which, as we will see, can be useful in dealing with LG distributions.

In order to follow the link of the distribution of $LG(t; \delta, \alpha, \lambda, n)$ to splines, provided by Lemma 2.2, let us first note that, for integer values of the parameters $\alpha_0 t > 0, \ldots, \alpha_n t > 0$, the density, $f_{B(t)}(x)$, of the random variable, $B(t)$, coincides with a polynomial B-spline. This is an important probabilistic interpretation of B-splines, established independently by Ignatov and Kaishev (1985, 1989a) and Karlin et al. (1986). In order to give a more precise formulation of this result, which will be used in the sequel, let us recall some background properties of polynomial B-splines. Let $\delta = \{\delta_0, \ldots, \delta_n\}$ denote a set of pairwise distinct real values $\delta_0 < \ldots < \delta_n$, called knots of the spline and denote by $\alpha = \{\alpha_0, \ldots, \alpha_n\}$ the set of their corresponding integer-valued multiplicities. The multiplicity $\alpha_i = 1, 2, \ldots$ equals the number of repetitions of the knot δ_i in the set of possibly coincident knots of the spline. Let us recall that the polynomial B-spline $M\left(x; \underset{\alpha_0,}{\delta_0}, \ldots, \underset{\alpha_n}{\delta_n}\right)$ of order $r = \alpha_0 + \ldots + \alpha_n - 1$ (degree $r - 1$) with knots $\delta = \{\delta_0, \ldots, \delta_n\}$ of multiplicities $\alpha = \{\alpha_0, \ldots, \alpha_n\}$ coincides with a polynomial of degree $r - 1$ between its adjacent (pairwise distinct) knots and is defined as the r-th order divided difference of the function $f(y) = r(y - x)_+^{r-1}$, i.e.,

$$M\left(x; \underset{\alpha_0,}{\delta_0}, \ldots, \underset{\alpha_n}{\delta_n}\right) = \left[\underset{\alpha_0,}{\delta_0}, \ldots, \underset{\alpha_n}{\delta_n}\right] f(y)$$

where the notation $\left[\underset{\alpha_0,}{\delta_0}, \ldots, \underset{\alpha_n}{\delta_n}\right] f(y)$ means that the arguments $\delta_0, \ldots, \delta_n$ of the divided difference are repeated $\alpha_0, \ldots, \alpha_n$ times, respectively.

The B-spline $M\left(x; \underset{\alpha_0,}{\delta_0}, \ldots, \underset{\alpha_n}{\delta_n}\right)$ has the following explicit representations. If knots are pair-wise distinct and their multiplicities $\alpha_0 = 1, \ldots, \alpha_n = 1$, then

$$M\left(x; \underset{1,}{\delta_0}, \ldots, \underset{1}{\delta_n}\right) = n \sum_{i=0}^{n} (\delta_i - x)_+^{n-1} \Big/ \prod_{j=0, j \neq i}^{n} (\delta_i - \delta_j)^{\, n-1}$$

If some of the knots coincide, i.e., $\alpha_0 \geq 1, \ldots, \alpha_n \geq 1$, then

$$M\left(x; \underset{\alpha_0,}{\delta_0}, \ldots, \underset{\alpha_n}{\delta_n}\right) = \sum_{i=0}^{n} D^{\alpha_i - 1} \xi_i(\delta_i) \,/\, (\alpha_i - 1)!,$$

where $\xi_i(y) = r(y - x)_+^{r-1} / \prod_{j=0, j \neq i}^{n} (y - \delta_j)^{\, \alpha_j}$ and $D^{\alpha_i - 1}$ denotes the $(\alpha_i - 1)$-th derivative.

The following theorem, due to Ignatov and Kaishev (1985, 1989a) establishes an important probabilistic interpretation of polynomial B-splines which we will use to study the distributional properties of LG processes.

Theorem 2.3 *(Ignatov and Kaishev 1985, 1989a). The polynomial B-spline* $M\left(x; \underset{\alpha_0,\ \ldots,\ \alpha_n}{\delta_0, \ldots, \delta_n}\right)$ *of degree* $\alpha_0 + \ldots + \alpha_n - 2$ *coincides with the probability density function,* $f_B(x)$*, of the random variable*

$$B = \delta_0\theta_0 + \ldots + \delta_n\theta_n,$$

where the random variables $\theta_0, \ldots, \theta_n$ *have joint Dirichlet distribution with parameters,* $\alpha_0, \ldots, \alpha_n$*, i.e.,* $(\theta_0, \ldots, \theta_n) \in \mathfrak{D}(\alpha_0, \ldots \alpha_n)$*.*

Let us note that the Dirichlet parameters $\alpha_0, \ldots, \alpha_n$, may in general take real values. In this case the density, $f_B(x)$, has been viewed by Ignatov and Kaishev (1987, 1988, 1989b) as a generalized B-spline. Independently, Karlin et al. (1986) have also considered similar generalization of B-splines. Later, such densities have been named Dirichlet splines (see Neuman 1994; zu Castell 2002). Here and thereafter, we will use the two terms, generalized B-splines and Dirichlet splines interchangeably. For consistency with the polynomial B-spline notation, we will alternatively denote, $f_B(x)$ as $M_g\left(x; \underset{\alpha_0,\ \ldots,\ \alpha_n}{\delta_0, \ldots, \delta_n}\right)$, to stress its interpretation as a generalized B-spline, i.e. a Dirichlet spline. We will make use of the following properties of generalized B-splines.

Let $\delta = \{\delta_0, \ldots, \delta_n\}$ be the set of pairwise distinct knots, $\delta_i \in \mathbb{R}$, and $\alpha = \{\alpha_0, \ldots, \alpha_n\}$ be the set of (positive real) multiplicities of $\delta = \{\delta_0, \ldots, \delta_n\}$. Denote by $\hat{\alpha}_i$ the integer part of α_i, and by $\bar{\alpha}_i = \alpha_i - \hat{\alpha}_i$ its fractional part. Without loss of generality, assume that $\bar{\alpha}_i > 0$, $i = 0, \ldots, k$, and that $\bar{\alpha}_i = 0$, $i = k+1, \ldots, k+m$, $(n = k+m)$.

The generalized B-spline can be expressed as the following divided difference (see Ignatov and Kaishev 1988, 1989b)

$$M_g\left(x; \underset{\alpha_0,\ \ldots,\ \alpha_n}{\delta_0, \ldots, \delta_n}\right) = \begin{cases} \left[\underset{\hat{\alpha}_0,\ \ldots,\ \hat{\alpha}_k,\ \alpha_{k+1},\ \ldots,\ \alpha_{k+m}}{\delta_0, \ldots, \delta_k,\ \delta_{k+1}, \ldots, \delta_{k+m}}\right] H(u), & \text{if } x \in [[\mathcal{D}]]; \\ 0, & \text{otherwise.} \end{cases}$$

where

$$H(u)=\frac{\Gamma\left(\alpha_0+\ldots+\alpha_{k+m}\right)}{\Gamma(l-1)\Gamma\left(\bar{\alpha}_0\right)\ldots\Gamma\left(\bar{\alpha}_k\right)}\int_{S_k}\left(u-x+\sum_{i=0}^{k}\left(\delta_i-u\right)y_i\right)_+^{l-2}y_0^{\bar{\alpha}_0-1}\ldots y_k^{\bar{\alpha}_k-1}dy_0\ldots dy_k,$$

$l=\sum_{i=0}^{k+m}\hat{\alpha}_i$, $(l\geq 2)$, $S_k=\{(y_0,\ldots,y_k):0\leq y_i, i=0,\ldots,k, y_0+\ldots+y_k\leq 1\}$ and $\mathcal{D}$ is the set of all δ_i's for which $\hat{\alpha}_i\geq 1$, $[[\mathcal{D}]]$ denotes the convex hull of $\mathcal{D}$.

The numerical evaluation of generalized B-splines is facilitated by their representation in terms of classical polynomial B-splines, due to Kaishev (1991). For further properties of generalized B-splines (i.e. Dirichlet splines) we refer to Neuman (1994) and zu Castell (2002).

We can now formulate and prove the following proposition which expresses the density of $LG(t)$ in terms of Dirichlet splines.

Proposition 2.4 *For fixed t, the density, $f_{LG(t)}(x)$, of $LG(t;\delta,\alpha,\lambda,n)$ is given by*

(2.6)

$$f_{LG(t)}(x)=\int_0^{+\infty}\frac{\lambda^{(\alpha_0+\ldots+\alpha_n)t}}{\Gamma\left(\left(\alpha_0+\ldots+\alpha_n\right)t\right)}y^{(\alpha_0+\ldots+\alpha_n)t-2}e^{-\lambda y}M_g\left(\frac{x}{y};\begin{matrix}\delta_0,\ldots,\delta_n\\ \alpha_0 t,\ \ldots,\ \alpha_n t\end{matrix}\right)dy,$$

where $M_g\left(\frac{x}{y};\begin{matrix}\delta_0,\ldots,\delta_n\\ \alpha_0 t,\ \ldots,\ \alpha_n t\end{matrix}\right)$, is a Dirichlet spline with knots, $\delta_0,\ldots,\delta_n$, of (real) multiplicities, $\alpha_0 t,\ldots,\alpha_n t$.

Proof: By Lemma 2.2, we have that $LG(t;\delta,\alpha,\lambda,n)$ is expressed as a product of two independent random variables with known densities. More precisely, the random variable, $\Gamma(t)=\sum_{i=0}^{n}G_i\left(t;\alpha_i,\lambda\right)$, is gamma distributed with parameters $\left(\alpha_0+\ldots+\alpha_n\right)t$ and λ, i.e., $\Gamma(t)\sim\text{Gamma}\left(\left(\alpha_0+\ldots+\alpha_n\right)t,\lambda\right)$, whereas, by Theorem 2.3, the density $f_{B(t)}(x)$, of the random variable, $B(t)$, coincides with a generalized B-spline. We will denote the density of $\Gamma(t)$, as $f_{\Gamma(t)}(x)$.

Thus, we have

$$\begin{aligned}f_{LG(t)}(x)&=\frac{d}{dx}P\left(B(t)\Gamma(t)\leq x\right)=\frac{d}{dx}P\left(B(t)\leq\frac{x}{\Gamma(t)}\right)\\&=\frac{d}{dx}\int_0^{+\infty}P\left(B(t)\leq\frac{x}{y}\right)f_{\Gamma(t)}(y)dy\\&=\int_0^{+\infty}f_{B(t)}\left(\frac{x}{y}\right)f_{\Gamma(t)}(y)\frac{1}{y}dy.\end{aligned}$$

The result now follows, noting that

$$f_{\Gamma(t)}(y) = \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)t}}{\Gamma\left((\alpha_0+\ldots+\alpha_n)\,t\right)} y^{(\alpha_0+\ldots+\alpha_n)t-1} e^{-\lambda y}, \tag{2.7}$$

and that $f_{B(t)}(x/y)$ coincides with a generalized B-spline, $M_g\left(\frac{x}{y};\, \underset{\alpha_0 t,}{\delta_0}, \ldots, \underset{\alpha_n t}{\delta_n}\right)$.
□

Several properties of the process $LG(t;\delta,\alpha,\lambda,n)$ easily follow from Proposition 2.4.

Corollary 2.5 *If $\alpha_i t$ are integer valued, the density, $f_{LG(t)}(x)$, of $LG(t;\delta,\alpha,\lambda,n)$ is given by*

(2.8)

$$f_{LG(t)}(x) = \int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)t}}{\Gamma\left((\alpha_0+\ldots+\alpha_n)\,t\right)} y^{(\alpha_0+\ldots+\alpha_n)t-2} e^{-\lambda y} M\left(\frac{x}{y};\, \underset{\alpha_0 t,}{\delta_0}, \ldots, \underset{\alpha_n t}{\delta_n}\right) dy,$$

where $M\left(\frac{x}{y};\, \underset{\alpha_0 t,}{\delta_0}, \ldots, \underset{\alpha_n t}{\delta_n}\right)$, is a polynomial B-spline with knots, $\delta_0,\ldots,\delta_n$, of multiplicities, $\alpha_0 t,\ldots,\alpha_n t$.

Corollary 2.6 *The density of the increments, $LG(t+h;\delta,\alpha,\lambda,n)-LG(t;\delta,\alpha,\lambda,n)$, $h>0$ is given by*

$$\int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)h}}{\Gamma\left((\alpha_0+\ldots+\alpha_n)\,h\right)} y^{(\alpha_0+\ldots+\alpha_n)h-2} e^{-\lambda y} M_g\left(\frac{x}{y};\, \underset{\alpha_0 h,}{\delta_0}, \ldots, \underset{\alpha_n h}{\delta_n}\right) dy.$$

Proof: We have

$$LG(t+h;\delta,\alpha,\lambda,n) - LG(t;\delta,\alpha,\lambda,n) =$$
$$\delta_0\left[G_0\left(t+h;\alpha_0,\lambda\right) - G_0\left(t;\alpha_0,\lambda\right)\right] + \ldots + \delta_n\left[G_n\left(t+h;\alpha_n,\lambda\right) - G_n\left(t;\alpha_n,\lambda\right)\right],$$

which, for fixed t and $h>0$, is a linear combination of gamma variates $g_i = \left[G_i\left(t+h;\alpha_i,\lambda\right) - G_i\left(t;\alpha_i,\lambda\right)\right]$ with density

$$f_{g_i}\left(x;\alpha_i,\lambda,h\right) = \frac{\lambda^{\alpha_i h}}{\Gamma\left(\alpha_i h\right)} x^{\alpha_i h-1} e^{-\lambda x}.$$

Obviously, for fixed t and $h > 0$ we can write

$$LG(t+h;\delta,\alpha,\lambda,n) - LG(t;\delta,\alpha,\lambda,n) = \left[(\delta_0 g_0 + \ldots + \delta_n g_n) / \sum_{i=0}^{n} g_i\right]\left[\sum_{i=0}^{n} g_i\right],$$

which is in the form of (2.5). Hence, the Corollary follows from Lemma 2.2 and Theorem 2.3. □

We conclude this section by noting that the following proposition which is a direct consequence of the scaling property of the gamma distribution provides an alternative way of expressing the underlying LG distribution, as a linear combination of $n+1$ gamma variates with different shape and scale parameters.

Proposition 2.7 *The process $LG(t;\delta,\alpha,\lambda,n)$ admits the representation*

(2.9)

$$LG(t;\delta,\alpha,\lambda,n) = sgn(\delta_0)\, G_0\left(t;\alpha_0,\lambda/\left|\delta_0\right|\right) + \ldots + sgn(\delta_n)\, G_n\left(t;\alpha_n,\lambda/\left|\delta_n\right|\right).$$

It has to be noted that extensive literature exists which deals with the distribution underlying (2.9), in the special case when $\text{sgn}(\delta_j) = +1$, $j = 0,\ldots,n$. In the latter case, an explicit formula for the density of $LG(t;\delta,\alpha,\lambda,n)$ when t is fixed, $t > 0$, is given by Moschopoulos (1985).

2.2 The Variance Gamma and the Bilateral Gamma special cases

New expressions for the density of the Variance Gamma, $VG(t;\theta,\sigma,\nu)$ and the Bilateral Gamma processes directly follow from their LG representation, Proposition 2.4 and Corollary 2.5. We have

Corollary 2.8 *For fixed t, the density, $f_{VG(t)}(x;\theta,\sigma,\nu)$, of the Variance Gamma process, $VG(t;\theta,\sigma,\nu)$ is given by*

(2.10) $f_{VG(t)}(x;\theta,\sigma,\nu) =$

$$\int_0^{+\infty} \frac{1}{\Gamma(2t/\nu)} y^{2t/\nu-2} e^{-y} M_g\left(\frac{x}{y}; -\frac{\sqrt{\theta^2+2\,\sigma^2/\nu}-\theta}{\underset{t/\nu}{2}}\nu, \frac{\sqrt{\theta^2+2\,\sigma^2/\nu}+\theta}{\underset{t/\nu}{2}}\nu\right) dy,$$

where $M_g\left(\frac{x}{y};\underset{t/\nu}{\cdot},\underset{t/\nu}{\cdot}\right)$ *coincides with a classical polynomial B-spline of degree* $\frac{2t}{\nu}-2$ *if* $\frac{t}{\nu}$ *is integer.*

Recall that a different expression for the density $f_{VG(t)}(x;\theta,\sigma,\nu)$, has been given by Madan et al. (1998) as follows

$$f_{VG(t)}(x;\theta,\sigma,\nu)=\int_0^{+\infty}\frac{1}{\sigma\sqrt{2\pi y}}e^{-\frac{(x-\theta y)^2}{2\sigma^2 y}}\frac{1}{\nu^{\frac{t}{\nu}}\Gamma(t/\nu)}y^{t/\nu-1}e^{-\frac{y}{\nu}}dy.$$

For the density of the Bilateral Gamma process we have the following result.

Corollary 2.9 *For fixed,* $t>0$*, the density,* $f_{BG(t)}(x)$*, of the Bilateral Gamma process,* $BG\left(t;\alpha^+,\lambda^+,\alpha^-,\lambda^-\right)$*, is given by*

(2.11)

$$f_{BG(t)}(x)=\int_0^{+\infty}\frac{1}{\Gamma\left(\left(\alpha^-+\alpha^+\right)t\right)}y^{\left(\alpha^-+\alpha^+\right)t-2}e^{-y}M_g\left(\frac{x}{y};\underset{\alpha^-t}{-\left(\lambda^-\right)^{-1}},\underset{\alpha^+t}{\left(\lambda^+\right)^{-1}}\right)dy.$$

where $M_g\left(\frac{x}{y};\underset{\alpha^-t}{\cdot},\underset{\alpha^+t}{\cdot}\right)$ *coincides with a polynomial B-spline of degree* $\alpha^-t+\alpha^+t-2$ *if the parameters,* α^-t,α^+t*, are integer.*

For comparison with (2.11), for $t=1$, the density, $f_{BG(t)}(x)$ given by Küchler and Tappe (2008) is

$$f_{BG}(x)=\frac{\left(\lambda^+\right)^{\alpha^+}\left(\lambda^-\right)^{\alpha^-}}{\left(\lambda^++\lambda^-\right)^{\frac{\alpha^++\alpha^-}{2}}\Gamma\left(\alpha^+\right)}x^{\frac{\alpha_0+\alpha_1}{2}-1}e^{-\frac{x\left(\lambda^+-\lambda^-\right)}{2}}W_{\frac{\left(\alpha^+-\alpha^-\right)}{2},\frac{\left(\alpha^++\alpha^--1\right)}{2}}\left(x\left(\lambda^++\lambda^-\right)\right),$$

where $W_{\omega,\mu}(z)$ is the Whittaker function defined as

$$W_{\omega,\mu}(z)=\frac{z^\omega e^{-z/2}}{\Gamma(\mu-\omega+1/2)}\int_0^{+\infty}t^{\mu-\omega-1/2}e^{-t}\left(1+\frac{t}{z}\right)^{\mu+\omega-1/2}dt$$

for $\mu-\omega>-\frac{1}{2}$.

In conclusion, let us note that expressions (2.6), (2.8) (2.10) and (2.11), involving generalized or polynomial B-splines, are numerically appealing, due to the recurrent computation of polynomial B-splines (see de Boor 1976) and the cubature formula for generalized B-splines (i.e. Dirichlet splines) in terms of polynomial B-splines, due to Kaishev (1991).

2.3 The Lévy triplet and related properties

As known, (see e.g. Cont and Tankov 2004, section 3.4), the characteristic triplet, (γ, A, κ), i.e., the Lévy triplet of a (multivariate) Lévy process, comprised by, a (real) vector γ, a positive definite (covariance) matrix A and a positive measure κ, related to its Lévy-Itô decomposition, uniquely determines its distribution. Following the Lévy-Khinchin representation formula, it is possible to express the characteristic function, $\phi_{LG}(z) = \mathbb{E}\left[e^{izLG(t)}\right]$, of a LG process, in terms of its corresponding Lévy triplet(γ, A, κ) and deduce some path-wise properties. The following Proposition gives the Lévy triplet of an LG process.

Proposition 2.10 *$LG(t;\delta,\alpha,\lambda,n)$ is a Lévy process with characteristic triplet $(\gamma, 0, \kappa_{LG})$, where the Lévy measure $\kappa_{LG}(dx)$ is given by*

$$\kappa_{LG}(dx) = \left(\sum_{i\in D_-} \frac{\alpha_i e^{-\lambda\frac{|x|}{|\delta_i|}}}{|x|}\mathbf{1}_{x<0} + \sum_{i\in D_+} \frac{\alpha_i e^{-\lambda\frac{x}{\delta_i}}}{x}\mathbf{1}_{x>0}\right) dx \tag{2.12}$$

with $D_- = \{i \in I : \delta_i < 0\}$, $D_+ = \{i \in I : \delta_i \geq 0\}$, $I = \{0, \ldots, n\}$ and

$$\gamma = \frac{1}{\lambda}\left(\sum_{i\in D_+} \alpha_i\delta_i\left(1 - e^{-\frac{\lambda}{\delta_i}}\right) - \sum_{i\in D_-} \alpha_i\,|\delta_i|\left(1 - e^{-\frac{\lambda}{|\delta_i|}}\right)\right) < \infty. \tag{2.13}$$

Proof: Since $LG(t;\delta,\alpha,\lambda,n)$ is defined as a linear combination of the gamma processes, $G_i\,(t;\alpha_i,\lambda)$, $i = 0,\ldots,n$, which are Lévy processes, $LG(t;\delta,\alpha,\lambda,n)$ is also a Lévy process (see, e.g. Theorem 4.1 of Cont and Tankov 2004). Expression (2.12) for the Lévy measure $\kappa_{LG}dx$ follows from the additivity property of the Lévy measure (see e.g. Proposition 5.3, Theorem 4.1 and Example 4.1 of Cont and Tankov 2004) and representation (2.9), noting that the Lévy measure of the process $\beta_i(t) = \operatorname{sgn}(\delta_i)\,G_i\,(t;\alpha_i,\lambda/\,|\delta_i|)$ is

$$\kappa_{\beta_i}(dx) = \left(\frac{\alpha_i \exp\left(-\lambda|x|\,/|\delta_i|\,\right)}{|x|}\mathbf{1}_{\{x<0,\delta_i<0\}} + \frac{\alpha_i \exp\left(-\lambda x/\delta_i\right)}{x}\mathbf{1}_{x>0,\delta_i>0}\right) dx.$$

Clearly, there is no Brownian motion component in the definition of $LG(t;\delta,\alpha,\lambda,n)$, hence the second parameter of the characteristic triplet is 0.

Due to the fact that, the drift parameter of the gamma process, $G_i(t; \alpha_i, \lambda)$, is 0, from Corollary 3.1 of Cont and Tankov (2004), we have that

$$\gamma = \int_{|x|\leq 1} x\kappa_{LG}(dx), \tag{2.14}$$

and by substituting (2.12) in (2.14) we obtain (2.13). □

From the analytical properties of its characteristic triplet, $(\gamma, 0, \kappa_{LG})$, it is straightforward to deduce that the LG process has piece-wise constant trajectories, is a process of finite variation and infinite activity (i.e., may have infinitely many small jumps). These path-wise properties are in fact inherited from the gamma processes, underlying the definition of an LG process (see Definition 2.1). Let us also note that, the LG process offers extended flexibility in controlling its Lévy measure, $\kappa_{LG}(dx)$, compared to the VG and BG processes. In the case of an LG process, one can manipulate its parameters and alter its Lévy measure, $\kappa_{LG}(dx)$ so that the distribution of the size of only the positive jumps, or only the negative jumps changes, (see Fig. 1, right panel). This may often be desirable in practical applications, but is not possible for the VG process. Changing the VG parameters, θ, σ and ν, affects both the positive and the negative parts of its Lévy measure, which is illustrated in Fig. 1, left panel.

As known (see Proposition 3.18 of Cont and Tankov 2004), the exponent of a (univariate) Lévy process with characteristic triplet (γ, A, κ) is a martingale, if and only if $\int_{|x|\geq 1} e^x \kappa(dx) < \infty$ and

$$\frac{A}{2} + \gamma + \int_{-\infty}^{+\infty} \left(e^x - 1 - x\mathbf{1}_{|x|\leq 1}\right) \kappa_{LG}(dx) = 0. \tag{2.15}$$

Based on this result, Propositions 2.11 and 2.12 establish conditions for the exponent of an LG process to be a martingale, a property which is important in financial applications.

Proposition 2.11 *Given* $n \geq 1$, $\alpha_i > 0$, $\int_{|x|\geq 1} e^x \kappa_{LG}(dx) < \infty$ *if and only if* $\lambda > \max_{i\in D_+} \{\delta_i\}$.

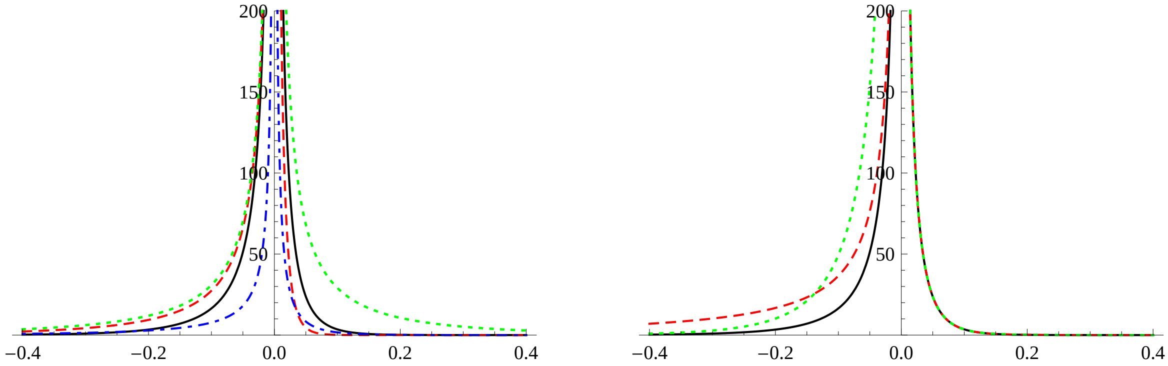

Figure 1: *left panel: Lévy measure of a VG process for the following four sets of parameters $\theta = -0.29$, $\sigma = 0.19$, $\nu = 0.25$ (solid line); $\theta = -0.99$, $\sigma = 0.19$, $\nu = 0.25$, (dashed); $\theta = -0.29$, $\sigma = 0.99$, $\nu = 0.25$ (dotted) and $\theta = -0.29$, $\sigma = 0.19$, $\nu = 0.95$ (dot-dashed); right panel: The LG Lévy measure for $\lambda = 1$, and the following three sets of parameters $\delta_0 = -0.11$, $\delta_1 = 0.04$, $\alpha_0 = \alpha_1 = 3.99$ (solid line); $\delta_0 = -1.13$, $\delta_1 = 0.04$, $\alpha_0 = \alpha_1 = 3.99$ (dashed); and $\delta_0 = -0.11$, $\delta_1 = 0.04$, $\alpha_0 = 11.98$, $\alpha_1 = 3.99$ (dotted);*

Proof: It can be directly verified, substituting $\kappa_{LG}(dx)$ from (2.12) that, for $x > 0$,

$$\int_{|x|\geq 1} e^x \kappa_{LG}(dx) = \sum_{i\in D_+} \int_1^{+\infty} \frac{\alpha_i \exp\left[-x\left(\lambda/\delta_i - 1\right)\right]}{x} dx. \tag{2.16}$$

We have that,

$$\int_1^{+\infty} \frac{\alpha_i \exp\left[-x\left(\lambda/\delta_i - 1\right)\right]}{x} dx = \begin{cases} \alpha_i E_1\left(\lambda/\delta_i - 1\right) < \infty, & \text{if} \quad \lambda > \delta_i \\ \text{diverges}, & \text{otherwise} \end{cases},$$

where $E_1\left(\lambda/\delta_i - 1\right)$ denotes the *Exponential Integral* (defined in section 5.1.4 of Abramowitz and Stegun 1972), evaluated at $\lambda/\delta_i - 1 > 0$, from where it can be seen that, the sum in (2.16) will converge, if and only if $\lambda > \max_{i\in D_+}\{\delta_i\}$.

Similarly, it can be verified that, for $x < 0$, we have that

$$\int_{|x|\geq 1} e^x \kappa_{LG}(dx) = \sum_{i\in D_-} \int_1^{+\infty} \frac{\alpha_i \exp\left[-|x|\left(\lambda/\left|\delta_i\right| + 1\right)\right]}{|x|} dx = -\sum_{i\in D_-} \alpha_i \mathrm{Ei}\left(\lambda/\delta_i - 1\right) < \infty,$$

where $\mathrm{Ei}\left(\lambda/\delta_i - 1\right)$ denotes the Exponential Integral function (defined in section 5.1.2 of Abramowitz and Stegun 1972), evaluated at $\lambda/\delta_i - 1 < 0$, from where it can be seen that in order for the sum in (2.16) to converge, no additional conditions on the parameters λ and $\delta_i$need to be imposed. $\square$

Proposition 2.12 *The exponential LG process,* $\exp(LG(t;\delta,\alpha,\lambda,n))$ *is a (local) martingale, if and only if*

$$\sum_{i\in D_+}\alpha_i\ln\left(1-\frac{\delta_i}{\lambda}\right)=-\sum_{i\in D_-}\alpha_i\ln\left(1+\frac{|\delta_i|}{\lambda}\right). \tag{2.17}$$

Proof: From (2.15) and (2.14), noting that, for a LG process, $A=0$, it follows that $\exp(LG(t;\delta,\alpha,\lambda,n))$ is a martingale if and only if

$$\int_{-\infty}^{+\infty}(e^x-1)\,\kappa_{LG}(dx)=0. \tag{2.18}$$

We have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(e^x-1)\,\kappa_{LG}(dx)= \\
&\int_{-\infty}^{0}x\sum_{j=0}^{\infty}\frac{x^j}{(j+1)!}\sum_{i\in D_-}\alpha_i\frac{e^{-\lambda\frac{|x|}{|\delta_i|}}}{|x|}dx+\int_0^{+\infty}x\sum_{j=0}^{\infty}\frac{x^j}{(j+1)!}\sum_{i\in D_+}\alpha_i\frac{e^{-\lambda\frac{x}{\delta_i}}}{x}dx= \\
&\sum_{i\in D_+}\alpha_i\sum_{j=0}^{\infty}\int_0^{+\infty}\frac{x^j}{(j+1)!}e^{-\lambda\frac{x}{\delta_i}}dx-\sum_{i\in D_-}\alpha_i\sum_{j=0}^{\infty}\int_{-\infty}^{0}\frac{x^j}{(j+1)!}e^{\lambda\frac{x}{|\delta_i|}}dx= \\
&\sum_{i\in D_+}\alpha_i\sum_{j=0}^{\infty}\left(\frac{\delta_i}{\lambda}\right)^{j+1}\frac{1}{j+1}-\sum_{i\in D_-}\alpha_i\sum_{j=0}^{\infty}(-1)^j\left(\frac{|\delta_i|}{\lambda}\right)^{j+1}\frac{1}{j+1}= \\
&-\sum_{i\in D_+}\alpha_i\ln\left(1-\frac{\delta_i}{\lambda}\right)-\sum_{i\in D_-}\alpha_i\ln\left(1+\frac{|\delta_i|}{\lambda}\right),
\end{aligned} \tag{2.19}$$

where $\ln\left(1-\frac{\delta_i}{\lambda}\right)$ is well defined, given that, $\lambda>\max_{i\in D_+}\{\delta_i\}$, as required by Proposition 2.11. From (2.19) it follows that (2.18) is satisfied if and only if (2.17) is satisfied. $\square$

Remark 2.13 *It is not difficult to see that the right-hand side of (2.19) vanishes* *if $n \geq 1$, the sets D^- and D^+ have equal cardinality and*

$$-\alpha_{i^-} \ln\left(1 + \frac{|\delta_{i^-}|}{\lambda}\right) = \alpha_{i^+} \ln\left(1 - \frac{\delta_{i^+}}{\lambda}\right),$$

where $i^- \in D^-$ and $i^+ \in D^+$, which holds true if for example

$$-\alpha_{i^-} = \alpha_{i^+}, |\delta_{i^-}| = -\frac{\delta_{i^+}\lambda}{\delta_{i^+} - \lambda} \text{ and } \lambda > \max_{i \in D_+} \{\delta_i\}. \tag{2.20}$$

Hence, for a fixed $n \geq 1$, one can always chose a set $\{\delta_i\}$, $i \in D_+$ and select values, λ, $|\delta_{i^-}|$, $i^- \in D_-$, and α_i, $i \in I$, according to (2.20), so that (2.19) vanishes and the model, $\exp(LG(t; \delta, \alpha, \lambda, n))$, is a (local) martingale. In contrast, it is not difficult to see from the LG representation, (2.2), of a VG process that, there does not exist a set of VG parameters, (θ, σ, ν) for which $\exp(VG(t; \theta, \sigma, \nu))$ is a martingale. However, in applications, one will typically consider VG processes with an additional linear drift in which case it is of course possible to turn the exponential VG into a martingale, as illustrated by (2.22) for the LG model with linear drift.

Remark 2.14 *Let us note that the statements*

(1) $\exp(LG(t; \delta, \alpha, \lambda, n))$ *is a martingale*

(2) $\exp(LG(t; \delta, \alpha, \lambda, n))$ *is a local martingale*

are equivalent. The equivalence, (1) $\Leftrightarrow$ (2) follows from Proposition 1.47, Chapter 1 of Jacod and Shiryaev (2003) and Lemma 4.4 (part 3.) of Kallsen (2000). Therefore, if condition (2.17) is satisfied, the $\exp(LG(t; \delta, \alpha, \lambda, n))$, *is in fact a local martingale. Furthermore, noting that* $\exp(LG(t; \delta, \alpha, \lambda, n))$ *is neither increasing nor decreasing, it directly follows from Theorem 4.6 (a) of Cherny and Shiryaev (2002) (see also Jakubenas 2002) that the exponential LG model,* $\exp(LG(t; \delta, \alpha, \lambda, n))$ *satisfies the "No Free Lunch with Vanishing Risk" property, introduced by Delbaen and Schachermayer (1994) as a continuous time analogue of the no arbitrage condition.*

We conclude this section by briefly indicating that the (univariate) LG process can be used for modelling asset price dynamics. Define the (risk-neutral) asset price process, $S(t)$ as

$$S(t) = S(0)\exp((r - q + \omega)t + LG(t;\delta,\alpha,\lambda,n)) \tag{2.21}$$

where r - the (constant) risk-free rate, q - the dividend yield, and the constant ω is chosen so that $\mathbb{E}(S(t)) = S(0)\exp((r-q)t)$, i.e.

$$\omega = \sum_{i\in D_+} \alpha_i \log\left(1 - \frac{\delta_i}{\lambda}\right) + \sum_{i\in D_-} \alpha_i \log\left(1 + \frac{|\delta_i|}{\lambda}\right) \tag{2.22}$$

which follows from Proposition 2.12. We therefore require $\lambda > \max_{i\in D_+}\{\delta_i\}$. Note that, in the special case of the $VG(t;\theta,\sigma,\nu)$ process, (2.22) yields

$$\omega = \frac{1}{\nu}\log\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right),$$

where $1 > \frac{\sqrt{\theta^2+2\,\sigma^2/\nu}+\theta}{2}\nu$ (which implies $1 > \left(\theta + \sigma^2/\,2\right)\nu$).

The model given by (2.21) can be used in (exotic) option pricing and pricing participating life insurance contracts. Due to volume limitations, details of how this is done are outside the scope of this paper and will appear elsewhere.

3 Multivariate LG processes

In what follows we will consider the multivariate generalization of univariate LG processes, defined in section 2, which, as we will illustrate in section 4, can be very useful in modelling the joint dynamics of possibly dependent prices of multiple assets. We start with the following definition.

Definition 3.1 *Define the multivariate LG process,* $\boldsymbol{LG}(t) = (LG_1(t),\dots,LG_s(t))'$, $(s \geq 1)$ *as*

$$\begin{aligned} LG_1(t) &= \delta_{1,0}G_0(t;\alpha_0,\lambda) + \dots + \delta_{1,n}G_n(t;\alpha_n,\lambda) \\ &\vdots \\ LG_s(t) &= \delta_{s,0}G_0(t;\alpha_0,\lambda) + \dots + \delta_{s,n}G_n(t;\alpha_n,\lambda), \end{aligned} \tag{3.1}$$

where $\boldsymbol{\delta_j} = (\delta_{1,j},\dots,\delta_{s,j})'$, $\boldsymbol{\delta_j} \in \mathbb{R}^s$, $j = 0,\dots,n$, *are pairwise distinct,* $n \geq s$,

$\lambda > 0$, $\alpha = \{\alpha_0, \ldots, \alpha_n\}$, $\alpha_j > 0$, $j = 0, \ldots, n$ and $G_j(t; \alpha_j, \lambda)$, $j = 0, \ldots, n$ are independent Gamma processes defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

From Definition 3.1, it can be seen that all coordinates, $\mathrm{LG}_i(t)$, $i = 1, \ldots, s$ jump together. It should also be noted that simulation of a multivariate LG trajectory is straightforward since it requires simulating Gamma sample paths which can be done very efficiently, applying the Dirichlet bridge sampling method, recently proposed by Kaishev and Dimitrova (2009).

Before proceeding further, we will need to introduce the following notation. For a given set $A \subset \mathbb{R}^s$, $\mathbf{1}_A(x)$, $[[A]]$, $\mathrm{vol}_s(A)$, $\dim(A)$ denotes the indicator function, the closed convex hull, the s-dimensional Lebesgue measure and the dimension respectively. By $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, $\ldots$ we denote elements (vectors) in the Euclidean space $\mathbb{R}^s$ $(s \geq 1)$, i.e., $\boldsymbol{x} = (x_1, \ldots, x_s)'$ where, $'$, means transposition and we use subscripts to index vectors, i.e., $\boldsymbol{x}_j = (x_{1,j}, \ldots, x_{s,j})'$, $j = 0, 1, \ldots$. We denote by $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{s} x_i y_i$ the inner product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^s$.

3.1 Distributional properties

In what follows, we study distributional properties of multivariate LG processes and establish their relation to multivariate splines. For the purpose we will need to introduce multivariate B-splines, known also as simplex splines. A simplex spline is a multivariate version of the univariate polynomial B-spline defined in Section 2.1. Simplex splines, were first introduced by de Boor (1976) as follows.

Definition 3.2 *(de Boor, 1976) Let $\mathfrak{S} = [[\boldsymbol{y}_0, \ldots, \boldsymbol{y}_r]]$ be any r-simplex in $\mathbb{R}^r$, $\mathbb{R}^r = \mathbb{R}^s \times \mathbb{R}^{r-s}$, such that $\boldsymbol{y}_j|_{\mathbb{R}^s} = \boldsymbol{\delta}_j$, $j = 0, \ldots, r$, i.e., the first s coordinates of $\boldsymbol{y}_j$ agree with the vector $\boldsymbol{\delta}_j \in \mathbb{R}^s$, $s \geq 1$. The multivariate (simplex) spline $M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r)$ is defined as*

$$M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r) = vol_{r-s}\left(\{\boldsymbol{u} \in \mathfrak{S} : \boldsymbol{u}|_{\mathbb{R}^s} = \boldsymbol{x}\}\right) / vol_r(\mathfrak{S}).$$

Note that Definition 3.2 allows for coalescent knots, $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r$ of which say, $n+1 < r+1$ knots, $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ may be pairwise distinct with corresponding multiplicities $\alpha_0, \ldots, \alpha_n$, $\alpha_0 + \ldots + \alpha_m = r+1$. If there are $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ pairwise distinct knots with multiplicities $\alpha_0, \ldots, \alpha_n$, $M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r)$, will be alternatively denoted as $M\left(\boldsymbol{x};\underset{\alpha_0,\ \ldots,\ \alpha_n}{\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_n}\right)$, $v \in \mathbb{R}^s$ and also as $M\left(x_1,\ldots,x_s;\underset{\alpha_0,\ \ldots,\ \alpha_n}{\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_n}\right)$.

The simplex spline, $M(\boldsymbol{x};\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r)$ is a piecewise polynomial of total degree not exceeding $r-s$ with $r-s-1$ continuous derivatives when the knots, $\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r$ are in general position. The knots, $\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r$, are said to be in general position if for $j=1,\ldots,s$ and for arbitrary, different indexes $0 \le i_1,\ldots,i_{j+1} \le r$, we have

$$\det\begin{pmatrix} 1 & \delta_{1,i_1} & \ldots & \delta_{j,i_1} \\ 1 & \delta_{1,i_2} & \ldots & \delta_{j,i_2} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \delta_{1,i_{j+1}} & \ldots & \delta_{j,i_{j+1}} \end{pmatrix} \neq 0.$$

The numerical evaluation of multivariate simplex splines is facilitated by the following recurrence relation, due to Micchelli (1980)

$$M(\boldsymbol{x};\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r) = \frac{r}{r-s}\sum_{j=0}^{r}\lambda_j M(\boldsymbol{x};\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_{j-1},\boldsymbol{\delta}_{j+1},\ldots,\boldsymbol{\delta}_r), \tag{3.2}$$

whenever $r > s$ and the numbers, $\lambda_j \in \mathbb{R}$, are such that, $\boldsymbol{x} = \sum_{j=0}^{r}\lambda_j\boldsymbol{\delta}_j$, $\sum_{j=0}^{r}\lambda_j = 1$.

For further properties of simplex splines see e.g., Neamtu (2001), Cohen et al. (2001) and Prautzsch (2002).

We will now recall that simplex splines have a nice probabilistic interpretation established independently by Karlin et al. (1986) and Ignatov and Kaishev (1985, 1989a) which we will exploit in studying the properties of multivariate LG processes. Given the set of knots $\Delta = \{\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r\}$, $\boldsymbol{\delta}_j = (\delta_{1,j},\ldots,\delta_{s,j})'$, $\boldsymbol{\delta}_j \in \mathbb{R}^s$, $j=0,\ldots,r$, consider the random vector $\boldsymbol{B} = (B_1,\ldots,B_s)'$, defined by

$$\boldsymbol{B} = \boldsymbol{\delta}_0\theta_0 + \ldots + \boldsymbol{\delta}_r\theta_r, \tag{3.3}$$

with coordinates $B_i = \delta_{i,0}\theta_0 + \ldots + \delta_{i,r}\theta_r$, $i=1,\ldots,s$, where the random vector $\boldsymbol{\theta} = (\theta_0,\ldots,\theta_r)'$, is Dirichlet distributed with parameters $\alpha = \{1,\ldots,1\}$, i.e. $(\theta_0,\ldots,\theta_r) \in \mathfrak{D}(1)$.

It will be convenient to view the vectors $\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r$ as points in $\mathbb{R}^s$, $s \ge 1$. Note that in (3.3), we allow some of the points $\boldsymbol{\delta}_0,\ldots,\boldsymbol{\delta}_r$ to coalesce. Let us assume that only $n+1$ of them are pairwise distinct, say $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$, each repeated with multiplicity $\alpha_0, \ldots, \alpha_n$, $\alpha_0 + \ldots + \alpha_n = r+1$. Then, given the set of distinct knot parameters, $\Delta = \{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n\}$, following a well known property of the Dirichlet distribution (see e.g. Wilks 1962), the random vector $\boldsymbol{B} = (B_1, \ldots, B_s)'$, defined by (3.3), can be rewritten as

$$\boldsymbol{B} = \boldsymbol{\delta}_0 \theta_0 + \ldots + \boldsymbol{\delta}_n \theta_n, \tag{3.4}$$

with coordinates $B_i = \delta_{i,0}\theta_0 + \ldots + \delta_{i,n}\theta_n$, $i = 1, \ldots, s$, where the random vector $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_n)'$, is Dirichlet distributed with parameters $\alpha = \{\alpha_0, \ldots, \alpha_n\}$, i.e., $(\theta_0, \ldots, \theta_n) \in \mathfrak{D}(\alpha_0, \ldots, \alpha_n)$.

Assume also that the parameters α, Δ, and n, are such that the distribution of the linear transformation $\boldsymbol{B}$ and its marginal distributions exist and are non-degenerate. Denote by $f_{\boldsymbol{B}}(\boldsymbol{x})$ the density of $\boldsymbol{B}$. The following result establishes the probabilistic interpretation of simplex splines.

Theorem 3.3 *(Ignatov and Kaishev 1985, 1989a). Let $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ be fixed pairwise distinct vectors in $\mathbb{R}^s$, $n \geq s$, with dimension* $\dim([[\{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n\}]]) = s$, *then the density $f_{\boldsymbol{B}}(\boldsymbol{x})$ with respect to the s-dimensional Lebesgue measure of the random vector $\boldsymbol{B}$, defined as in (3.4), coincides with the simplex spline*

$$M\left(\boldsymbol{x}; \underset{\alpha_0,}{\boldsymbol{\delta}_0}, \underset{\ldots,}{\ldots}, \underset{\alpha_n}{\boldsymbol{\delta}_n}\right)$$

with knots $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ having (integer) multiplicities, $\alpha_0, \ldots, \alpha_n$.

As in the univariate case, the Dirichlet parameters $\alpha_0, \ldots, \alpha_n$, may in general take real values. In this case the density, $f_{\boldsymbol{B}}(\boldsymbol{x})$, has been viewed by Ignatov and Kaishev (1987, 1988, 1989b) as a multivariate generalized B-spline i.e., as multivariate Dirichlet spline. Independently, Karlin et al. (1986) have also considered similar generalization of multivariate simplex splines. For some further properties of multivariate Dirichlet splines see Karlin et al. (1986), Ignatov and Kaishev (1987, 1988, 1989b) and Neuman (1994).

The following proposition gives for fixed $t > 0$ an expression for the joint density of the multivariate LG process in terms of multivariate Dirichlet splines.

Proposition 3.4 *Let $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$, $\boldsymbol{\delta}_j \in \mathbb{R}^s$, $n \geq s$, be pairwise distinct and let* $\dim [[\{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n\}]] = s$, *then for fixed t the density of $\boldsymbol{LG}(t)$ is*

(3.5) $f_{\boldsymbol{LG}(t)}(x_1, \ldots, x_s) =$

$$\int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)t}}{\Gamma\left((\alpha_0 + \ldots + \alpha_n)\, t\right)} y^{(\alpha_0+\ldots+\alpha_n)t-(s+1)} e^{-\lambda y} M_g \left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}; \underset{\alpha_0 t,\ \ldots,\ \alpha_n t}{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n} \right) dy,$$

where $\Gamma(\cdot)$ *is the gamma function and* $M_g \left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}; \underset{\alpha_0 t,\ \ldots,\ \alpha_n t}{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n} \right)$ *is a multivariate Dirichlet spline with knots* $\Delta = \{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n\}$, *of multiplicities* $\{\alpha_0 t, \ldots \alpha_n t\}$.

Proof: For fixed $t > 0$, we have that the multivariate LG process can be represented as

$$\boldsymbol{LG}(t; \Delta, \alpha, \lambda, n) = \boldsymbol{B}(t)\Gamma(t) \tag{3.6}$$

where $\boldsymbol{B}(t)$ is defined as in (3.4) and has a joint density $f_{\boldsymbol{B}(t)}(\boldsymbol{x})$, which, by Theorem 3.3, coincides with a generalized B-spline, and where the random variable, $\Gamma(t) = \sum_{i=0}^{n} G_i\,(t; \alpha_i, \lambda)$, independent of $\boldsymbol{B}(t)$, is gamma distributed with parameters $(\alpha_0 + \ldots + \alpha_n)\, t$ and λ. Thus, we have

$$\begin{aligned}
f_{\boldsymbol{LG}(t)}(x_1, \ldots, x_s) &= \frac{\partial}{\partial x_1 \ldots \partial x_s} P\left(B_1(t) \times \Gamma(t) \leq x_1, \ldots, B_s(t) \times \Gamma(t) \leq x_s\right) \\
&= \frac{\partial}{\partial x_1 \ldots \partial x_s} P\left(B_1(t) \leq \frac{x_1}{\Gamma(t)}, \ldots, B_s(t) \leq \frac{x_s}{\Gamma(t)}\right) \\
&= \int_0^{+\infty} \frac{\partial}{\partial x_1 \ldots \partial x_s} P\left(B_1(t) \leq \frac{x_1}{y}, \ldots, B_s(t) \leq \frac{x_s}{y}\right) f_{\Gamma(t)}(y)dy \\
&= \int_0^{+\infty} f_{\boldsymbol{B}(t)}\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}\right) f_{\Gamma(t)}(y) \frac{1}{y^s} dy.
\end{aligned}$$

The result now follows, in view of (2.7) and noting that, by Theorem 3.3, $f_{\boldsymbol{B}(t)}\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}\right)$ coincides with a multivariate Dirichlet spline, $M_g \left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}; \underset{\alpha_0 t,\ \ldots,\ \alpha_n t}{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n} \right)$.
□

In case $\alpha_i t$, $i = 0, \ldots, n$, are integers then $M_g(\cdot)$ is a classical multivariate polynomial simplex spline, given by Definition 3.2 and its evaluation can be successfully performed using e.g. Michelli's recurrence (3.2). When the multiplicities $\alpha_i t$, $i = 0, \ldots, n$ are non-integer, to the best of our knowledge, the evaluation of multivariate Dirichlet splines has not been sufficiently explored. Recurrence formulas for the moments of multivariate Dirichlet splines and simplex splines have been established by Neuman (1994).

In order to provide some insight into the dependence properties of multivariate LG processes, next we give its underlying copula.

Proposition 3.5 *The copula $C_{LG}(u_1, \ldots, u_s)$, is given as*

$$(3.7) \quad C_{LG}(u_1, \ldots, u_s) = \int_{-\infty}^{F_{LG_1}^{-1}(u_1)} \ldots \int_{-\infty}^{F_{LG_s}^{-1}(u_s)} \int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)t}}{\Gamma((\alpha_0+\ldots+\alpha_n)t)} y^{(\alpha_0+\ldots+\alpha_n)t-(s+1)} e^{-\lambda y} M_g\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}; \underset{\alpha_0 t,\ \ldots,\ \alpha_n t}{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n}\right) dy dx_s \ldots dx_1,$$

where $u_i \in [0, 1]$, and

$$F_{LG_i}(x) = \int_{-\infty}^{x} \int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)t}}{\Gamma((\alpha_0+\ldots+\alpha_n)t)} y^{(\alpha_0+\ldots+\alpha_n)t-2} e^{-\lambda y} M_g\left(\frac{z}{y}; \underset{\alpha_0 t,\ \ldots,\ \alpha_n t}{\delta_{j,0}, \ldots, \delta_{j,n}}\right) dy dz,$$

$i = 1, \ldots, s$.

Proof: Expression (3.7) follows from Sklar's Theorem and expressions (3.5) and (2.6).□

Let us note that the LG copula $C_{LG}(u_1, \ldots, u_s)$ is related to the (new) class of the so- called Dirichlet (B-) spline copulas, introduced by Kaishev (2006b). Both B-spline copulas and LG copulas are quite flexible, and by controlling the knots, Δ, of the Dirichlet spline, and their multiplicities, α, one can model and reproduce a wide range of dependence structures arising in financial applications. This is illustrated in section 4, on the example of multivariate FX modelling. For further results and applications of B-spline copulas see Kaishev (2006b).

The next proposition gives the characteristic function of a multivariate LG process, which will be needed in order to develop a method of moments for estimating the LG parameters.

Proposition 3.6 *For fixed t, say $t = 1$, the characteristic function, $\phi(\boldsymbol{z})$, of* *the multivariate LG process,* $\boldsymbol{LG}(t)$*, is*

$$\phi_{\boldsymbol{LG}}(\boldsymbol{z}) = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - i\left(\boldsymbol{\delta_j} \cdot \boldsymbol{z}\right)} \right)^{\alpha_j} \tag{3.8}$$

where $\boldsymbol{\delta}_j = (\delta_{1,j}, \ldots, \delta_{s,j})' \in \mathbb{R}^s$*,* $j = 0, \ldots, n$ *,* $\boldsymbol{z} = (z_1, \ldots, z_s)' \in \mathbb{R}^s$ *and* $\lambda > 0$*.*

Proof: From Definition 3.1, for fixed t, say $t = 1$ and $\boldsymbol{z} = (z_1, \ldots, z_s)' \in \mathbb{R}^s$, it is directly seen that the characteristic function

$$\begin{aligned} \phi_{\boldsymbol{LG}}(\boldsymbol{z}) &= \mathbb{E}\left[e^{i(\boldsymbol{z} \cdot \boldsymbol{LG}(t))}\right] = \mathbb{E}\left[e^{i\left(\sum_{j=0}^{n} \boldsymbol{\delta}_j \cdot \boldsymbol{z} G_j(t;\alpha_j,\lambda)\right)}\right] \\ &= \prod_{j=0}^{n} \mathbb{E}\left[e^{i\boldsymbol{\delta}_j \cdot \boldsymbol{z} G_j(t;\alpha_j,\lambda)}\right] = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - i\left(\boldsymbol{\delta}_j \cdot \boldsymbol{z}\right)} \right)^{\alpha_j}, \end{aligned}$$

which completes the proof of the asserted expression for $\phi_{\boldsymbol{LG}}(\boldsymbol{z})$. □

In order to develop a method of moments for estimating the LG parameters, we will give here the moment generating function (mgf)

$$M_{\boldsymbol{LG}}(\boldsymbol{z}) = \mathbb{E}\left[e^{\boldsymbol{z} \cdot \boldsymbol{LG}(t)}\right] = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - \boldsymbol{\delta}_j \cdot \boldsymbol{z}} \right)^{\alpha_j},$$

and the cumulant generating function (cgf)

$$K_{\boldsymbol{LG}}(\boldsymbol{z}) = \log M_{\boldsymbol{LG}}(\boldsymbol{z}) = \sum_{j=0}^{n} -\alpha_j \log \left(1 - \frac{1}{\lambda} \boldsymbol{\delta}_j \cdot \boldsymbol{z} \right) \tag{3.9}$$

of the LG random vector.

3.2 LG parameter estimation: method of moments

There are two sources of difficulty related to estimating the parameters of a multivariate LG process, given an appropriate data set. Firstly, it is the curse of dimensionality, i.e., the dimension s may be very high which is typically the case in some credit risk modelling applications. Secondly, the underlying dependence pattern may be rather complex, requiring significant number, $n + 1 > s$ of knot parameters in each coordinate with corresponding multiplicities, and hence a large number of parameters overall.

Due to the latter difficulties, maximum likelihood estimation of LG parameters, utilizing expression (3.5), is not so straightforward and may require developing a special purpose optimization algorithm, using exhaustive numerical optimization methods such as, adapted simulated annealing. Development of such methods is outside our scope and will be a subject of another paper. Here we will develop a method of moments for the estimation of the LG parameters, which is simpler to implement and as will be illustrated in section 4, serves well the purpose of calibrating an FX model driven by a multivariate LG process.

In order to develop a method of moments for the estimation of LG parameters, we will need the following piece of general multivariate cumulant theory, provided by McCullagh (2008). In what follows we shall somewhat depart from the notation used so far and use the notationally convenient, Einstein's summation convention in order to denote scalar products. Thus, $z_r X_r$ denotes the linear combination $z_1 X_1 + \ldots + z_s X_s$, where X_i, $i = 1, \ldots, s$ are the coordinates of a random vector $\boldsymbol{X} = (X_1, \ldots, X_s)$. The square of a linear combination $(z_r X_r)^2 = (z_{r_1} X_{r_1})(z_{r_2} X_{r_2}) = z_{r_1} z_{r_2} X_{r_1} X_{r_2}$ is a sum of s^2 terms and for higher powers, $(z_r X_r)^l = z_{r_1} \ldots z_{r_2} X_{r_1} \ldots X_{r_l}$ is the sum of s^l terms. Following McCullagh (2008), we denote $\kappa_r = \mathbb{E}(X_r)$ the components of the mean vector, $\kappa_{r_1 r_2} = \mathbb{E}(X_{r_1} X_{r_2})$, $r_1, r_2 = 1, \ldots, s$ the components of the matrix of second moments, $\kappa_{r_1 r_2 r_3} = \mathbb{E}(X_{r_1} X_{r_2} X_{r_3})$, $r_1, r_2, r_3 = 1, \ldots, s$, the elements of the third moment matrix and so on, for the elements of the matrices of higher order moments.

The Taylor expansions of the moment generating function, $M_{\boldsymbol{X}}(\boldsymbol{z}) = \mathbb{E}\left[e^{z_r X_r}\right]$, and the cumulant generating function, $K_{\boldsymbol{X}}(\boldsymbol{z}) = \log M_{\boldsymbol{X}}(\boldsymbol{z})$, are then given as

$$M_{\boldsymbol{X}}(\boldsymbol{z}) = 1 + z_{r_1}\kappa_{r_1} + \frac{1}{2!} z_{r_1} z_{r_2} \kappa_{r_1 r_2} + \frac{1}{3!} z_{r_1} z_{r_2} z_{r_3} \kappa_{r_1 r_2 r_3} + \ldots$$

and

$$K_{\boldsymbol{X}}(\boldsymbol{z}) = z_{r_1}\kappa_{r_1} + \frac{1}{2!} z_{r_1} z_{r_2} \kappa_{r_1, r_2} + \frac{1}{3!} z_{r_1} z_{r_2} z_{r_3} \kappa_{r_1, r_2, r_3} + \ldots, \tag{3.10}$$

where κ_{r_1} denotes simultaneously first order moments and first order cumulants. The coefficients κ_{r_1, r_2}, $\kappa_{r_1, r_2, r_3}, \ldots$ in the expansion of $K_{\boldsymbol{X}}(\boldsymbol{z})$ are the corresponding second third and higher order cumulants. Note that, the latter are distinguished notationally from the corresponding moments, $\kappa_{r_1 r_2}$, $\kappa_{r_1 r_2 r_3}, \ldots$ by the commas separating subscripts. Equating the coefficients in the expansion of

$$K_{\boldsymbol{X}}(\boldsymbol{z}) = \log\left(1 + z_{r_1}\kappa_{r_1} + \frac{1}{2!} z_{r_1} z_{r_2}\kappa_{r_1 r_2} + \frac{1}{3!} z_{r_1} z_{r_2} z_{r_3}\kappa_{r_1 r_2 r_3} + \ldots\right)$$

to the corresponding coefficients in the expansion (3.10), it can be seen that each of the moments $\kappa_{r_1 r_2}$, $\kappa_{r_1 r_2 r_3}$, $\ldots$ can be expressed, as a sum over partitions of the subscripts, where each term in the sum is a product of cumulants, as follows

$$\kappa_{r_1 r_2} = \kappa_{r_1, r_2} + \kappa_{r_1}\kappa_{r_2} \tag{3.11}$$

$$\begin{aligned} \kappa_{r_1 r_2 r_3} &= \kappa_{r_1, r_2, r_3} + \kappa_{r_1, r_2}\kappa_{r_3} + \kappa_{r_1, r_3}\kappa_{r_2} + \kappa_{r_2, r_3}\kappa_{r_1} + \kappa_{r_1}\kappa_{r_2}\kappa_{r_3} \\ &= \kappa_{r_1, r_2, r_3} + \kappa_{r_1, r_2}\kappa_{r_3}[3] + \kappa_{r_1}\kappa_{r_2}\kappa_{r_3} \end{aligned} \tag{3.12}$$

(3.13)

$$\kappa_{r_1 r_2 r_3 r_4} = \kappa_{r_1, r_2, r_3, r_4} + \kappa_{r_1, r_2, r_3}\kappa_{r_4}[4] + \kappa_{r_1, r_2}\kappa_{r_3, r_4}[3] + \kappa_{r_1, r_2}\kappa_{r_3}\kappa_{r_4}[6] + \kappa_{r_1}\kappa_{r_2}\kappa_{r_3}\kappa_{r_4},$$

where the numbers in the square brackets indicate a sum over distinct partitions of the subscripts, having the same block sizes. Note that there are s equations $\kappa_{r_1} = \kappa_{r_1}$ relating the first order moments to the first order cumulants. In general, there are s^k equations for the moments of order $k = 1, 2, \ldots$ however, there are only $\binom{s+k-1}{k}$ distinct equations which coincides with the number of distinct moments of order k. Equations, (3.11) - (3.13), have been given by McCullagh (2008). Here, we further give the sets of equations, relating the fifth and the sixth order moments with the corresponding cumulants

$$\begin{aligned} \kappa_{r_1 r_2 r_3 r_4 r_5} &= \kappa_{r_1, r_2, r_3, r_4, r_5} + \kappa_{r_1, r_2, r_3, r_4}\kappa_{r_5}[5] + \kappa_{r_1, r_2, r_3}\kappa_{r_4, r_5}[10] + \\ &\kappa_{r_1, r_2, r_3}\kappa_{r_4}\kappa_{r_5}[5] + \kappa_{r_1, r_2}\kappa_{r_3, r_4}\kappa_{r_5}[15] + \kappa_{r_1, r_2}\kappa_{r_3}\kappa_{r_4}\kappa_{r_5}[10] + \kappa_{r_1}\kappa_{r_2}\kappa_{r_3}\kappa_{r_4}\kappa_{r_5}, \end{aligned} \tag{3.14}$$

(3.15)

$$\kappa_{r_1r_2r_3r_4r_5r_6} = \kappa_{r_1,r_2,r_3,r_4,r_5,r_6} + \kappa_{r_1,r_2,r_3,r_4,r_5}\kappa_{r_6}[6] + \kappa_{r_1,r_2,r_3,r_4}\kappa_{r_5,r_6}[15]+$$
$$\kappa_{r_1,r_2,r_3,r_4}\kappa_{r_5}\kappa_{r_6}[15]+\kappa_{r_1,r_2,r_3}\kappa_{r_4,r_5,r_6}[10]+\kappa_{r_1,r_2,r_3}\kappa_{r_4,r_5}\kappa_{r_6}[60]+\kappa_{r_1,r_2,r_3}\kappa_{r_4}\kappa_{r_5}\kappa_{r_6}[20]+$$
$$\kappa_{r_1,r_2}\kappa_{r_3,r_4}\kappa_{r_5,r_6}[15]+\kappa_{r_1,r_2}\kappa_{r_3,r_4}\kappa_{r_5}\kappa_{r_6}[45]+\kappa_{r_1,r_2}\kappa_{r_3}\kappa_{r_4}\kappa_{r_5}\kappa_{r_6}[15]+\kappa_{r_1}\kappa_{r_2}\kappa_{r_3}\kappa_{r_4}\kappa_{r_5}\kappa_{r_6}.$$

In what follows we will derive expressions for the cumulants of the random vector $\boldsymbol{LG}(t)$, in terms of the unknown parameters, Δ, α, λ and n. By substituting these expressions in the right-hand side of equations (3.11)-(3.15) and equating the theoretical moments, κ_{r_1}, $\kappa_{r_1r_2}$, $\kappa_{r_1r_2r_3}, \ldots$ to their corresponding empirical counterparts, one can solve the appropriate set of equations and obtain estimates of the unknown parameters. In what follows we will elaborate further on the details related to this method. The following proposition gives an expression for the cumulants of $\boldsymbol{LG}(t)$ in terms of the unknown parameters, Δ, α, λ and n.

Proposition 3.7 *The cumulant, $\kappa_{r_1,\ldots,r_w}$ of the random vector $\boldsymbol{LG}(t)$ is*

$$\kappa_{r_1,\ldots,r_w} = (w-1)!\sum_{j=0}^{n}\frac{\alpha_j}{\lambda^w}\delta_{r_1,j}\delta_{r_2,j}\ldots\delta_{r_w,j}, \tag{3.16}$$

where $w = 1, 2, \ldots,$ $r_i = 1,\ldots,s,$ $i = 1, \ldots, w$.

Proof: The cgf of the random vector, $\boldsymbol{LG}(t)$ can be expressed as in (3.10). On the other hand, from (3.9), we have

$$K_{\mathrm{LG}}(\boldsymbol{z}) = \sum_{j=0}^{n} \alpha_j \left(-\log\left(1 - \frac{1}{\lambda}\boldsymbol{\delta}_j \cdot \boldsymbol{z}\right)\right) \tag{3.17}$$

$$= \sum_{j=0}^{n} \alpha_j \left(\frac{1}{\lambda}\boldsymbol{\delta}_j \cdot \boldsymbol{z} + \frac{1}{2}\left(\frac{1}{\lambda}\boldsymbol{\delta}_j \cdot \boldsymbol{z}\right)^2 + \frac{1}{3}\left(\frac{1}{\lambda}\boldsymbol{\delta}_j \cdot \boldsymbol{z}\right)^3 + \dots\right)$$

$$= \sum_{j=0}^{n} \left(\left(\frac{\alpha_j}{\lambda}\boldsymbol{\delta}_j\right) \cdot \boldsymbol{z} + \frac{1}{2}\left(\left(\frac{\alpha_j^{1/2}}{\lambda}\boldsymbol{\delta}_j\right) \cdot \boldsymbol{z}\right)^2 + \frac{1}{3}\left(\left(\frac{\alpha_j^{1/3}}{\lambda}\boldsymbol{\delta}_j\right) \cdot \boldsymbol{z}\right)^3 + \dots\right)$$

$$= \sum_{j=0}^{n} \left(\left(\frac{\alpha_j}{\lambda}\right) \delta_{r_1,j} z_{r_1} + \frac{1}{2}\left(\frac{\alpha_j^{1/2}}{\lambda}\right) \delta_{r_1,j} z_{r_1} \left(\frac{\alpha_j^{1/2}}{\lambda}\right) \delta_{r_2,j} z_{r_2} + \right.$$

$$\left. \frac{1}{3}\left(\frac{\alpha_j^{1/3}}{\lambda}\right) \delta_{r_1,j} z_{r_1} \left(\frac{\alpha_j^{1/3}}{\lambda}\right) \delta_{r_2,j} z_2 \left(\frac{\alpha_j^{1/3}}{\lambda}\right) \delta_{r_3,j} z_{r_3} + \dots\right)$$

$$= z_{r_1} \sum_{j=0}^{n} \left(\frac{\alpha_j}{\lambda}\right) \delta_{r_1,j} + \frac{1}{2!} z_{r_1} z_{r_2} \sum_{j=0}^{n} \left(\frac{\alpha_j^{1/2}}{\lambda}\right) \delta_{r_1,j} \left(\frac{\alpha_j^{1/2}}{\lambda}\right) \delta_{r_2,j} +$$

$$\frac{1}{3!} z_{r_1} z_{r_2} z_{r_3} 2! \sum_{j=0}^{n} \left(\frac{\alpha_j^{1/3}}{\lambda}\right) \delta_{r_1,j} \left(\frac{\alpha_j^{1/3}}{\lambda}\right) \delta_{r_2,j} \left(\frac{\alpha_j^{1/3}}{\lambda}\right) \delta_{r_3,j} + \dots$$

Hence, comparing the coefficients of the corresponding terms $z_{r_1}, z_{r_1}z_{r_2}, z_{r_1}z_{r_2}z_{r_3}, \dots$ in (3.10) and (3.17) we have

$$\kappa_{r_1,\dots,r_w} = (w-1)! \sum_{j=0}^{n} \left(\frac{\alpha_j^{1/w}}{\lambda}\right) \delta_{r_1,j} \left(\frac{\alpha_j^{1/w}}{\lambda}\right) \delta_{r_2,j} \dots \left(\frac{\alpha_j^{1/w}}{\lambda}\right) \delta_{r_w,j}$$

$$= (w-1)! \sum_{j=0}^{n} \frac{\alpha_j}{\lambda^w} \delta_{r_1,j} \delta_{r_2,j} \dots \delta_{r_3,j},$$

which coincides with the asserted expression (3.16). □

We can now use (3.16) in order to express the cumulants on the right hand side of equations (3.11)-(3.15) and therefore, express the theoretical moments, $\kappa_{r_1 \dots r_w}$ in terms of the unknown parameters, Δ, α, λ and n. Then, equate the theoretical moments, $\kappa_{r_1 \dots r_w}$ to their empirical counterparts and solve with respect to Δ, α, λ, assuming n is appropriately chosen. It will be instructive to make the definition of the moments, $\kappa_{r_1 \dots r_w}$ a bit more precise.

Definition 3.8 *For $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)' \in \mathbb{N}^s$, define the moment,* $\mathbb{E}\left(X_1^{\beta_1} X_2^{\beta_2} \dots X_s^{\beta_s}\right)$, *of order* $|\boldsymbol{\beta}|$, $(|\boldsymbol{\beta}| = (\beta_1 + \ldots + \beta_s))$ *of the random vector* $\boldsymbol{LG}(t)$ *as*

$$\mathbb{E}\left(X_1^{\beta_1} X_2^{\beta_2} \ldots X_s^{\beta_s}\right) = \int_{\mathbb{R}^s} \boldsymbol{x}^{\boldsymbol{\beta}} f_{LG}(\boldsymbol{x}) d\boldsymbol{x} = \kappa_{r_1 \ldots r_{\beta_1} r_{\beta_1+1} \ldots r_{\beta_1+\beta_2} \ldots r_{\beta_1+\ldots+\beta_{s-1}+1} \ldots r_{|\boldsymbol{\beta}|}},$$

where $\boldsymbol{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \ldots x_s^{\beta_s}$ *and* $r_1 = \ldots = r_{\beta_1} = 1$, $r_{\beta_1+1} = \ldots = r_{\beta_1+\beta_2} = 2$, $\ldots$, $r_{\beta_1+\ldots+\beta_{s-1}+1} = \ldots = r_{|\boldsymbol{\beta}|} = s$.

The empirical moments of $\boldsymbol{LG}(t)$ can now be defined as follows.

Definition 3.9 *For* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)' \in \mathbb{N}^s$*, define the empirical moment* $\hat{\kappa}_{r_1 \ldots r_{|\boldsymbol{\beta}|}}$ *of order* $|\boldsymbol{\beta}|$*, as*

$$\hat{\kappa}_{r_1 \ldots r_{|\boldsymbol{\beta}|}} = \frac{1}{N} \sum_{l=1}^{N} x_{1,l}^{\beta_1} \ldots x_{s,l}^{\beta_s},$$

where $\{x_{1,l}, \ldots, x_{s,l}\}_{l=1}^N$ *is a sample of* N *i.i.d observations on* $\boldsymbol{LG}(t)$*.*

In order to estimate the parameters $(\Delta, \alpha, \lambda)$, based on $\{x_{1,l}, \ldots, x_{s,l}\}_{l=1}^N$, we apply the method of moments and solve the system

$$\left\{ \kappa_{r_1 \ldots r_{|\boldsymbol{\beta}|}}(\Delta, \alpha, \lambda) = \hat{\kappa}_{r_1 \ldots r_{|\boldsymbol{\beta}|}}, \quad |\boldsymbol{\beta}| = 1, 2, 3, \ldots \right. \tag{3.18}$$

with respect to $(\Delta, \alpha, \lambda)$.

Remark 3.10 *Note that there are* $\binom{s+k-1}{k}$ *distinct moments of order* $|\boldsymbol{\beta}| = k = 1, 2, 3, \ldots$ *. In an application, one would need to select the number of equations,* p*, in (3.18), to be equal to the number,* $(s+1) \times (n+1) + 1$*, of unknown parameters,* $(\Delta, \alpha, \lambda)$*, starting from moments of order 1 and increasing up to a maximum order* k^**, where* $k^* = \inf\left\{k : p \leq \sum_{j=1}^{k} \binom{s+j-1}{j}\right\}$.

The method of moments described here is illustrated in section 4 on the example of FX modelling.

4 Modelling the joint dynamics of exchange rates

Let $S_1(t)$, $S_2(t)$,..., $S_s(t)$, $t \geq 0$, be the exchange rates of a set of s currencies against a common reference currency. We are interested in modelling the joint dynamics of $S_1(t), S_2(t), \ldots, S_s(t)$, over a finite time interval $[0, T]$. We view $S_j(t)$, as the price of a risky asset with dynamics

$$S_j(t) = S_j(0) \exp\{X_j(t)\}, j = 1, \ldots, s,$$

where $X_j(t)$, $j = 1, \ldots, s$ are the coordinates of an appropriate s-variate stochastic process driving the joint FX dynamics. In what follows, we will compare the modelling results we obtain under two alternative choices for the processes $X_j(t), j = 1, \ldots, s$. Under the first choice, we implement the multivariate VG model with correlated Brownian motions and a common Gamma clock, proposed by Madan and Seneta (1990). We consider also its special non-correlated case considered by Luciano and Schoutens (2006). As an alternative, we implement the multivariate LG process, given in Definition 3.1.

In Fig. 2, we give the (historic) joint co-movement of the exchange rates of three currencies ($s = 3$), the Euro (EUR), the GB Pound (GBP) and the Japanese Yen (JPY) to the US Dollar as the reference (domestic) currency for the period $30.06.2008 - 30.06.2009$. As can be seen, examining Fig. 2 visually, there are different degrees of inter-dependence in the three FX trajectories. The exchange rates GBP/USD and EUR/USD exhibit stronger mutual correlation while at the same time, each of them is less correlated with the JPY/USD exchange rate. This is confirmed also if one analyses Fig. 3-4 which provide scatter plots and histograms of the corresponding log returns at unit time intervals, $\ln(S_j(t)/S_j(t-1)) = X_j(t) - X_j(t-1)$, $t = 1, 2, \ldots$. Examining Fig. 4 one can see that the (marginal) distributions of the corresponding (historic) daily log returns seem to exhibit heavier tails than in the normal case, which is a bit more expressed for the EUR/USD and JPY/USD. As has been noted by Daal and Madan (2005), a univariate VG density is an appropriate choice for fitting empirical FX data. The two dimensional scatter plots of the three pairs of log returns given in Fig. 3, show that the pair GBP/USD versus EUR/USD exhibits positive dependence with stronger upper tail dependence, the pair JPY/USD versus EUR/USD looks evenly scattered around the origin, while the pair JPY/USD versus GBP/USD looks somewhat negatively correlated.

First, we model the co-movement of the three FX rates, EUR/USD, GBP/USD and JPY/USD, indexed by $j = 1, 2, 3$ respectively, applying the multivariate VG

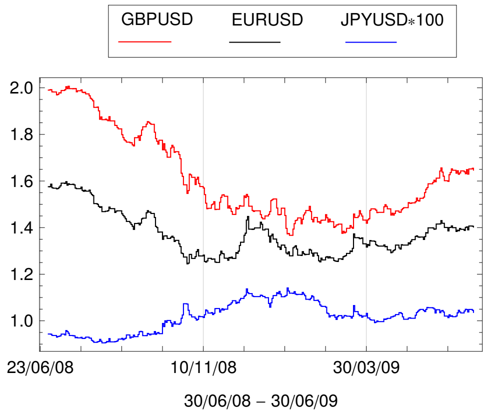

Figure 2: *Joint co-movement of the exchange rates of GBP/USD, EUR/USD, and JPY/USD, viewed from top to bottom.*

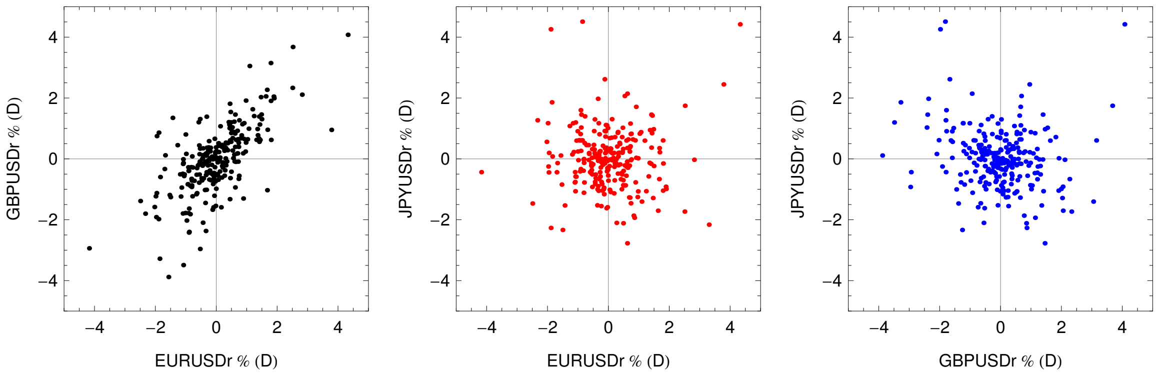

Figure 3: *Bilateral scatter plots of the empirical log returns EUR/USD, GBP/USD and JPY/USD.*

process, with correlated Brownian motions, and a common Gamma clock, proposed by Madan and Seneta (1990) and referred to by Deelstra and Petkovic (2010) as CCVG. For the vector of FX rates, we have the following specification of the CCVG model

$$S_j(t) = S_j(0) \exp\left\{m_j t + \theta_j G(t) + B_j\left(G(t)\right)\right\}, j = 1, 2, 3,$$

where $\boldsymbol{m} = (m_1, m_2, m_3)'$ are drift parameters, $B_j(G(t))$ are the coordinates of a 3-dimensional (correlated) Brownian motion with variance-covariance matrix $\Sigma = (\sigma_{ij})$, $G(t)$ is a common Gamma process with mean rate 1 and variance rate ν and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)'$ are the drift parameters of the VG part. There are 13 unknown parameters in total in this model and this is the maximum possible number of parameters for a three dimensional application ($s = 3$). Denote by $Z_j = \ln\left(S_j(t)/S_j(t-1)\right)$, the corresponding log returns. Then, applying the formula of total probability, conditioning on the gamma clock, (see Deelstra and Petkovic 2010) it is not difficult to derive the following expression for the density of the 3-dimensional CCVG

$$\text{(4.1)} \quad f_{Z_1,Z_2,Z_3}(\boldsymbol{z}) = \frac{2e^{(\boldsymbol{z}-\boldsymbol{m})'\Sigma^{-1}\boldsymbol{\theta}}}{(2\pi)^{\frac{3}{2}}|\Sigma|^{\frac{1}{2}}\nu^{\frac{1}{\nu}}\Gamma\left(\frac{1}{\nu}\right)} \left(\frac{(\boldsymbol{z}-\boldsymbol{m})'\Sigma^{-1}(\boldsymbol{z}-\boldsymbol{m})}{\frac{2}{\nu}+\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta}}\right)^{\frac{1}{2\nu}-\frac{3}{4}} K_{\frac{1}{\nu}-\frac{3}{2}}\left(\sqrt{\left((\boldsymbol{z}-\boldsymbol{m})'\Sigma^{-1}(\boldsymbol{z}-\boldsymbol{m})\right)\left(\frac{2}{\nu}+\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta}\right)}\right),$$

where $K_{\frac{1}{\nu}-\frac{3}{2}}()$ is the modified Bessel function of the second kind of order $\frac{1}{\nu}-\frac{3}{2}$. In order to illustrate the CCVG model, we first implement its special case of no correlation between the Brownian motions ($\Sigma = I_3$), referred to below as non-correlated CCVG. Such a special case has been considered by Luciano and Schoutens (2006). We have fixed $\nu = 1$ and estimated the unknown parameters, m_j, θ_j, σ_j $j = 1, 2, 3$, via marginal probabilities, as suggested by the authors. In Fig. 4, we give the histograms of (historic) daily log returns and fitted marginal $\mathrm{VG}_j\left(t; \theta_j, \sigma_j, \nu\right)$ densities and, as can be seen, they fit reasonably well the data.

In the upper panel of Fig. 5, we give the two dimensional scatter plots simulated from the corresponding fitted non-correlated CCVG. As can be seen from the upper panel of Fig. 5, comparing it with the corresponding empirical scatter plots of Fig. 3, the non-correlated CCVG model considered by Luciano and Schoutens (2006) fails to capture the underlying dependence in the data, especially for the EUR/USD, GBP/USD pair of currencies. In order to improve the

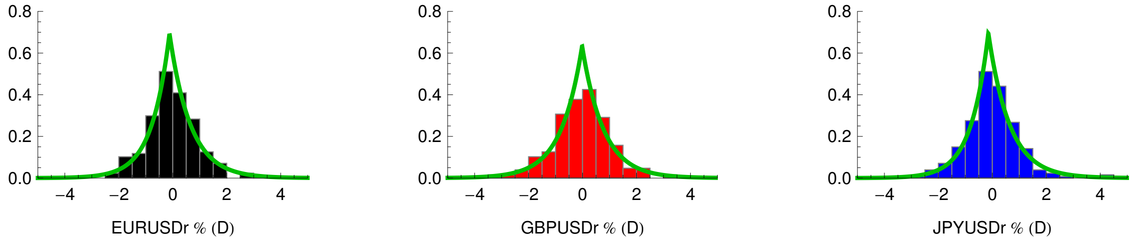

Figure 4: *Marginal VG densities fitted to (histograms of) historic daily log returns of the exchange rates of EUR/USD, GBP/USD and JPY/USD.*

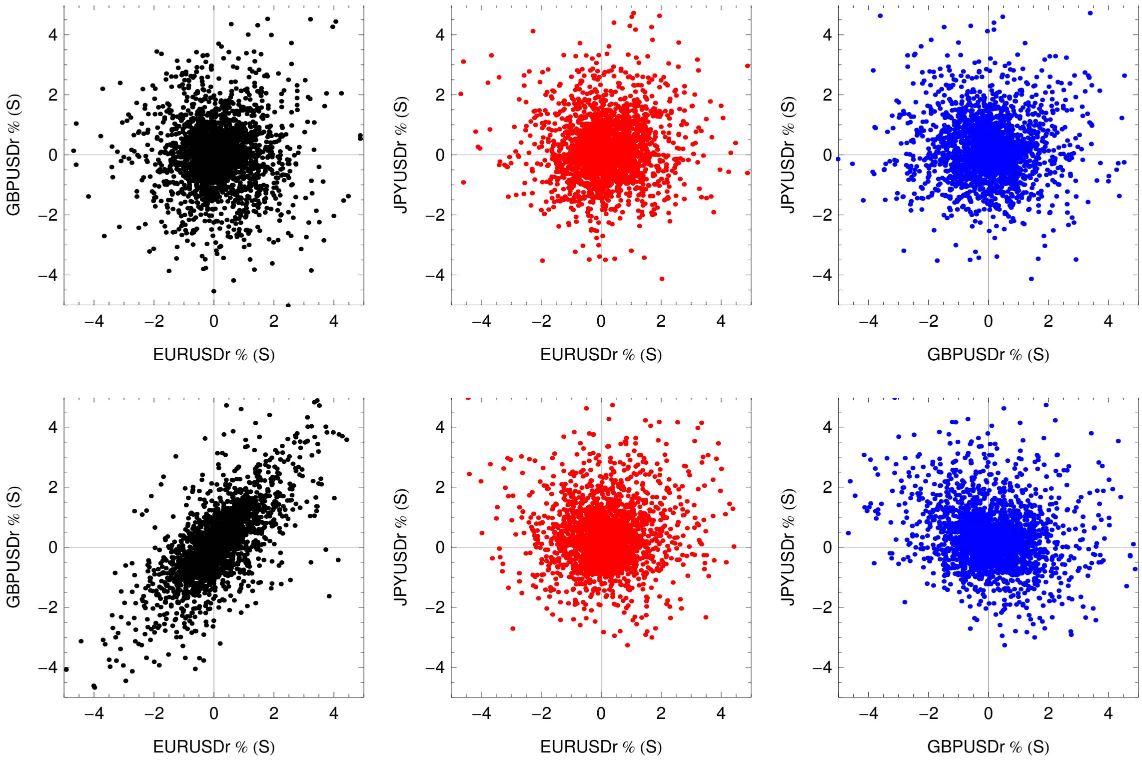

Figure 5: *Bilateral scatter plots of 2540 simulated three dimensional CCVG log returns EUR/USD, GBP/USD and JPY/USD, upper panel: from the 10 parameter, non-correlated CCVG; lower panel: from the 13 parameter, correlated CCVG.*

performance of the CCVG model we have implemented its correlated version, with 13 unknown parameters. Unfortunately, their estimation, applying the method of maximum likelihood using (4.1), turns out to be a formidable task, at least for the optimization routines of *Mathematica* 8, which we have attempted for the purpose. This is not surprising, given the complexity of the density in (4.1). Instead, we have used the method of moments on the marginal data in order to estimate the marginal parameters, $m_j, \theta_j, \sigma_{jj} j = 1, 2, 3$. As estimates of the covariances, σ_{12}, σ_{13} and σ_{23}, we have used the corresponding empirical covariances. Scatter plots illustrating the correlated CCVG model are given in the lower panel of Fig. 5. As can be seen, the introduction of the correlations σ_{12}, σ_{13} and σ_{23}, has substantially improved the performance of the CCVG, it captures well the existing positive correlation of the pair GBP/USD versus EUR/USD. However, it seems not to capture so well the somewhat stronger upper tail dependence in the data for that pair due to its conditional normality. The pair JPY/USD versus EUR/USD looks evenly scattered around the origin as does the data, while the pair JPY/USD versus GBP/USD is somewhat negatively correlated as the corresponding scatter plot of the data in Fig. 3 seems to suggest.

Alternatively, we model the co-movement of the three FX rates, EUR/USD, GBP/USD and JPY/USD, applying the proposed multivariate LG process, as follows

$$S_j(t) = S_j(0) \exp\left(\mathrm{LG}_j\left(t; \delta_{j,0}, \ldots, \delta_{j,n}, \alpha, \lambda, n\right)\right), \tag{4.2}$$

where $j = 1, 2, 3, n = 3, \lambda = 1, \alpha = \{1, 1, 1, 1\}$ and $\delta_{j,0}, \ldots, \delta_{j,n}$, are the 12 knot parameters. Note that 12 is the minimum possible number of knot parameters, since $n + 1 = 4$ is the minimum number of knots which span a volume in $\mathbb{R}^3$ ($s = 3$). For the purpose of estimating, $\delta_{j,0}, \ldots, \delta_{j,3}$, $j = 1, 2, 3$, we consider the joint distribution of the corresponding log returns, Z_j , $j = 1, 2, 3$, which is a three dimensional LG distribution. We have used the method of moments, developed in section 3.2, in order to estimate $\delta_{j,i}$, $j = 1, 2, 3$, $i = 0, 1, 2, 3$, by equating the first, second and third order theoretical moments, κ_{r_1}, $r_1 = 1, 2, 3$, $\kappa_{r_1 r_2}$, $r_1 = 1, 2, 3$, $r_2 = 1, 2, 3$, $r_1 < r_2$, $\kappa_{r_1 r_2 r_3}$, $r_1 = r_2 = r_3 = 1, 2, 3$, of the random vector (Z_1, Z_2, Z_3), given by (3.11) and (3.12), to their empirical counterparts, following (3.18). The LG marginal densities fit well the empirical data, which is illustrated in Fig. 6. In the upper panel of Fig. 7, we give the two

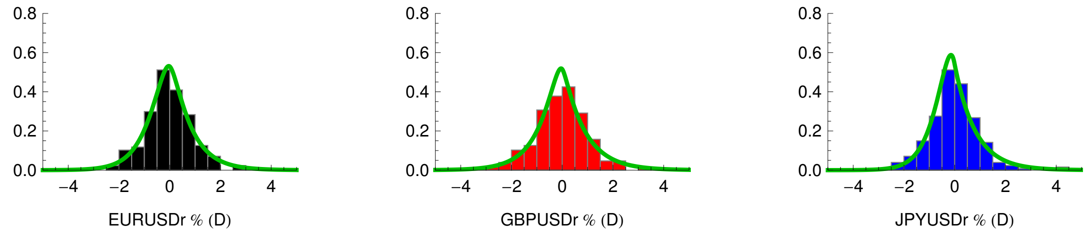

Figure 6: *Marginal LG densities fitted to (histograms of) historic daily log returns of the exchange rates of EUR/USD, GBP/USD and JPY/USD.*

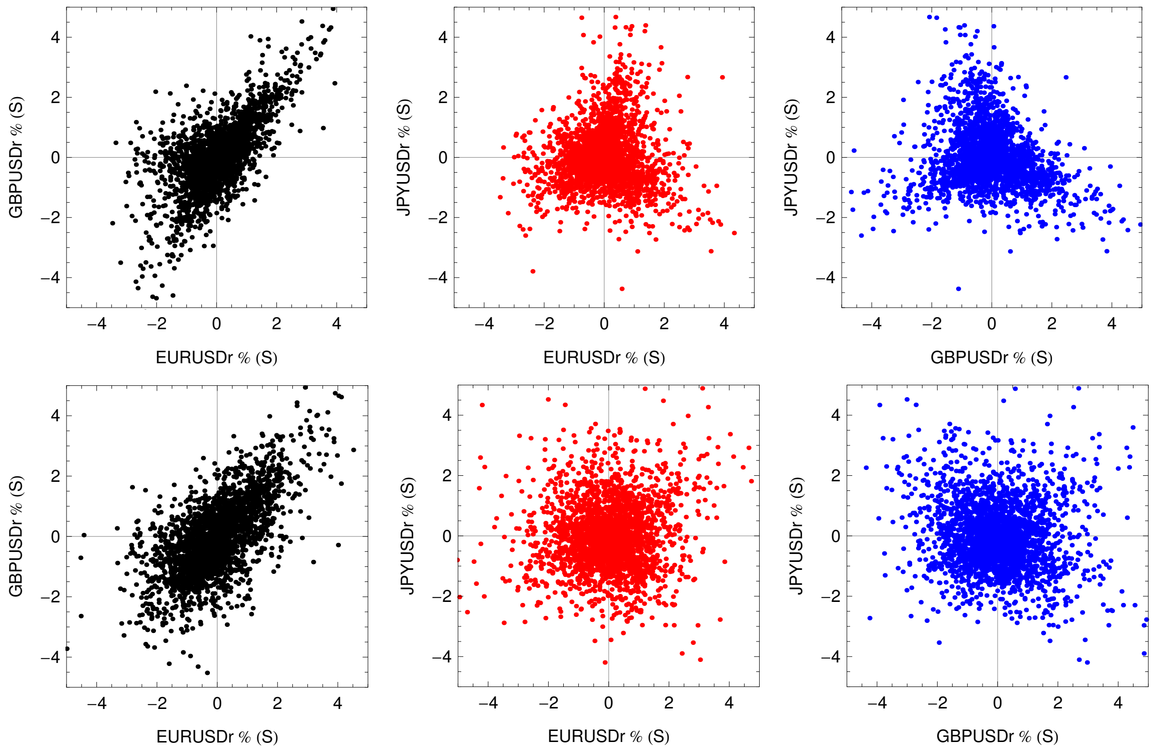

Figure 7: *Bilateral scatter plots of 2540 simulated LG log returns EUR/USD, GBP/USD and JPY/USD, upper panel: from a 12 parameter (4 knots) LG model, lower panel: from a 18 parameter (6 knots) LG model.*

dimensional scatter plots simulated from the corresponding fitted 12 parameter three dimensional LG distribution.

Comparing the scatter plots from the upper panel of Fig. 7 with the corresponding empirical scatter plots of Fig. 3, the 12 parameter, LG model captures the underlying dependence in the data, both for GBP/USD, EUR/USD and GBP/USD, JPY/USD pairs of exchange rates. As can be seen from the upper panel of Fig. 7, the multivariate LG vector can take any value in $\mathbb{R}^3$ but the scatter plots reveal a triangular shape inherited from the domain of the three dimensional Dirichlet spline, namely the triangular pyramid (tetrahedron) configuration defined by its four knots, δ_j, $j = 0, 1, 2, 3$ in $\mathbb{R}^3$. In contrast to the correlated CCVG model, for which the number of parameters is limited to a maximum of 13, it is possible to increase the number of LG parameters, say to six knots δ_j $j = 0, 1, 2, 3, 4, 5$, (which configures an octahedron in $\mathbb{R}^3$), and use the method of moments in order to get a better estimate of the underlying dependence structure. The improvement of the LG performance is illustrated in the lower panel of Fig. 7, where scatter plots from an 18 parameter (six knots) LG model are given.

In Fig. 8, we give sample paths simulated from the three dimensional LG model (4.2) which illustrates the higher correlation between the EUR/USD and GBP/USD which has also been empirically observed (see Fig. 2).

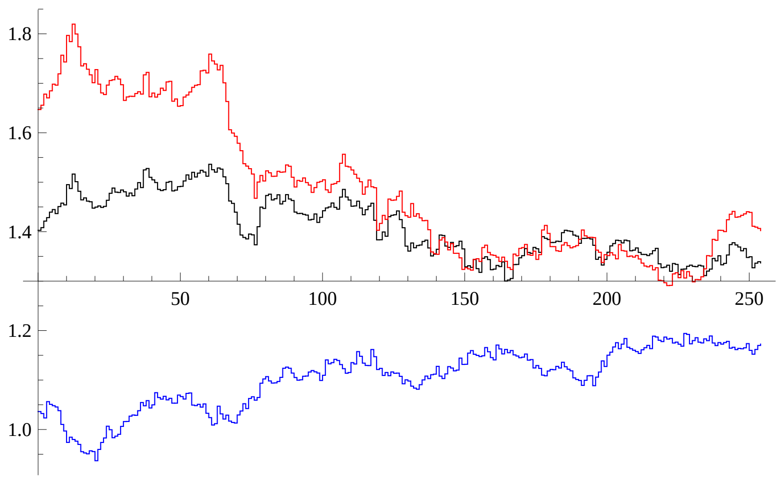

Figure 8: *Joint co-movement of the exchange rates of GBP/USD, EUR/USD and JPY/USD, viewed from top to bottom, simulated from the LG model.*

5 Comments and conclusions

The proposed (univariate) LG process is a more flexible generalization of the well-known VG process and the BG process since it allows to use any number of parameters according to the requirements of a particular application and control both the positive and the negative parts of the corresponding Lévy measure.

An enlightening link between the LG distribution, and (univariate) B-splines and Dirichlet splines is established and alternative formulas for the density of the VG and BG are given. The use of a LG process, as the driver of a stock price dynamics, in pricing exotic options and participating life insurance contracts is briefly indicated.

The proposed multivariate generalization of the LG process is very flexible, since it allows to incorporate any required number of parameters and to model complex dependence patterns between asset price processes. It is a competitive alternative to multivariate Lévy copulas and other multivariate generalizations of the VG process, based for instance, on a common random time change in a multivariate (correlated) Brownian motion.

We have also explored some of the properties of multivariate LG processes in terms of multivariate simplex B-splines and Dirichlet splines. In particular, we have given explicit expressions of the joint LG density and the underlying LG copula function in terms of Dirichlet splines, and also the LG characteristic, moment and cumulant generating functions. The latter have been used in section 3.2 to develop, a reasonably simple method of moments, based on their relation to cumulants, for the purpose of estimating the LG model parameters.

We have also illustrated the modelling power of a multivariate LG process on the example of FX modelling of the exchange rates of three currencies, the EUR the GBP and the JPY to the US Dollar. Results demonstrate that the LG process is a competitive alternative to the CCVG model with or without correlation in the Brownian motions. The LG model offers extended flexibility, which can prove important in handling more complex empirical dependence structures, e.g. exhibiting tail and directional dependencies.

Ongoing research is related to exploring market consistent LG parameter calibration and properties of the LG copula, which is a new promising member of the relatively limited family of multivariate copulas, richly enough parametrized so as to capture complex dependence patterns in truly multivariate financial and insurance applications. It is worth mentioning that yet another new class of copulas, related to the LG copulas, called Dirichlet (B-) spline copulas have been proposed and explored by Kaishev (2006b).

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