hep-th/0601204

Yukawa Couplings in Heterotic Standard Models

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Abstract

In this paper, we present a formalism for computing the Yukawa couplings in heterotic standard models. This is accomplished by calculating the relevant triple products of cohomology groups, leading to terms proportional to QHu, $Q\bar{H}d$, $LH\nu$ and $L\bar{H}e$ in the low energy superpotential. These interactions are subject to two very restrictive selection rules arising from the geometry of the Calabi-Yau manifold. We apply our formalism to the "minimal" heterotic standard model whose observable sector matter spectrum is exactly that of the MSSM. The non-vanishing Yukawa interactions are explicitly computed in this context. These interactions exhibit a texture rendering one out of the three quark/lepton families naturally light.

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Contents

1 Introduction

Obtaining non-vanishing Yukawa couplings is one of the most important issues in realistic superstring model building [1]. In this paper, we present a formalism for computing these terms and explicitly demonstrate, within an important class of $E_8 \times E_8$ superstring vacua, that non-vanishing Yukawa couplings are generated in the low energy effective theory.

In a series of papers [2–4] and [5], we presented a class of "heterotic standard model" vacua within the context of the $E_8 \times E_8$ heterotic superstring. The observable sector of a heterotic standard model vacuum is $N = 1$ supersymmetric and consists of a stable, holomorphic vector bundle, V, with structure group $SU(4)$ over an elliptically fibered

Calabi-Yau threefold, X, with a $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group. In [2–4], we gave non-trivial checks on the slope-stability of the vector bundle V. A rigorous proof of the stability of this bundle was presented in [6]. The vector bundle V in [5] is also slope-stable. This will be shown in detail in [7]. Each such bundle admits a gauge connection which, in conjunction with a Wilson line, spontaneously breaks the observable sector E_8 gauge group down to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ standard model group times an additional gauged $U(1)_{B-L}$ symmetry. The spectrum arises as the cohomology of the vector bundle V. For the vacuum presented in [5], the matter spectrum is found to be precisely that of the minimal supersymmetric standard model (MSSM). For this reason, we refer to [5] as the "minimal" heterotic standard model. The vacua presented in [2–4] also have the matter spectrum of the MSSM, with the exception of one additional pair of Higgs–Higgs conjugate superfields. These vacua contain *no exotic multiplets* and *no vector-like pairs of fields* with the exception of the Higgs pairs. They exist for both weak and strong string coupling. All previous attempts to find realistic particle physics vacua in superstring theories [8–24] have run into difficulties. These include predicting extra vector-like pairs of light fields, multiplets with exotic quantum numbers in the low energy spectrum, enhanced gauge symmetries and so on. Heterotic standard models avoid all of these problems. As for the hidden sector, there is no known obstruction to making it $N = 1$ supersymmetric as well, but we have not yet constructed the requisite hidden sector bundle. It is also unclear whether that is even phenomenologically desirable. In any case, in this paper we consider only the visible sector interactions.

Elliptically fibered Calabi-Yau threefolds with $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group were first constructed in [25–27] and [28, 29], respectively. More recently, the existence of elliptic Calabi-Yau threefolds with $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group was demonstrated and their classification given in [30]. In [31–34], methods for building stable, holomorphic vector bundles with arbitrary structure group in E_8 over simply-connected elliptic Calabi-Yau threefolds were introduced. These results were greatly expanded in a number of papers [25–27, 35–37] and then generalized to elliptically fibered Calabi-Yau threefolds with non-trivial fundamental group in [27–29, 38]. To obtain a realistic spectrum, it was found necessary to introduce a new method [25–29] for constructing vector bundles. This method, which consists of building the requisite bundles by "extension" from simpler, lower rank bundles, was used for manifolds with $\mathbb{Z}_2$ fundamental group in [27, 39–42] and in the heterotic standard model context in [30]. In [2–4, 41, 42], it was shown that to compute the complete low-energy spectrum of such vacua one must 1) evaluate the relevant sheaf cohomologies, 2) find the action of the finite fundamental group on these spaces and, finally, 3) tensor this with the action of the Wilson line on the associated representation. The low energy spectrum is the invariant cohomology subspaces under the resulting group action. This method was applied in [2–5] to compute the exact spectrum of all multiplets transforming non-trivially under the action of the low energy gauge group. The accompanying natural method of "doublet-triplet" splitting was also discussed. A formalism was presented in [43] that allows one to enumerate and describe the multiplets transforming trivially under the low energy gauge group, namely, the vector bundle moduli.

Using the above, one can construct a class of heterotic standard models and compute their entire low-energy spectrum. For example, using a $\mathbb{Z}_2$ Wilson line one can break a $SU(5)$ GUT group to the standard model gauge group. Heterotic vacua in this context were first computed in [41, 42]. This was recently refined in [44] to construct a realistic heterotic standard model with three chiral families of quarks/leptons and one pair of Higgs–Higgs conjugate fields. One can also use orbifold CFT to arrive at a minimal spectrum [45]. But for the purposes of this paper we will be interested $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines breaking a $Spin(10)$ GUT group to the standard model gauge group times $U(1)_{B-L}$. As mentioned previously, the observable sector spectrum consists exclusively of the three chiral families of quarks/leptons (each family with a right-handed neutrino), either one [5] or two [2–4] pairs of Higgs–Higgs conjugate fields and a small number of uncharged geometric and vector bundle moduli. However, finding the particle spectrum is far from the end of the story. To demonstrate that the particle physics in these vacua is realistic, one must construct the interactions of these fields in the low energy effective Lagrangian. These interactions occur in two distinct parts of the action. Recall that the matter part of an $N = 1$ supersymmetric Lagrangian is completely described in terms of two functions, the superpotential and the Kähler potential. Of these, the superpotential, being a "holomorphic" function of chiral superfields, is much more amenable to computation using methods of algebraic geometry. The superpotential itself is a sum of several different pieces, such as Higgs μ-terms and Yukawa couplings. In a recent paper [46], it was shown how to compute Higgs μ-terms in the superpotentials of heterotic standard models. In this paper, we continue our study of holomorphic interactions by presenting a formalism for computing Yukawa terms. We apply this method to calculate the Yukawa texture in the minimal heterotic standard model [5].

Specifically, we do the following. In Section 2, we review the relevant facts about the structure of heterotic standard model vacua in general and the minimal heterotic vacuum in particular. The formalism for computing the low energy spectrum is briefly discussed and we give the results for the minimal heterotic standard model vacuum. The structure of Yukawa terms are then analyzed and shown to occur as the product of three cohomology groups, two corresponding to the quark/lepton doublets (Q,L) and singlets (u,d,ν,e), and one corresponding to Higgs (H) and Higgs conjugate ($\bar{H}$) fields in the effective low energy theory. It follows that cubic terms of the form QHu, $Q\bar{H}d$, $LH\nu$ and $L\bar{H}e$ are potentially generated in the superpotential. Section 3 is devoted to discussing the first Leray spectral sequence, which is associated with the projection of the covering threefold $\widetilde{X}$ onto the base space B_2. The Leray decomposition of a sheaf cohomology group into (p, q) subspaces is discussed and applied to the cohomology spaces relevant to Yukawa terms. It is shown that the triple product is subject to a (p, q) selection rule which severely restricts the allowed non-vanishing terms. The second Leray decomposition, associated with the projection of the space B_2 onto its base $\mathbb{P}^1$, is presented in Section 4. The decomposition of any cohomology space into its $[s, t]$ subspaces is discussed and applied to cohomologies relevant to Yukawa terms. We show that Yukawa couplings are subject to yet another selection rule associated with the $[s, t]$ decomposition. Finally, it is demonstrated that the subspaces of cohomology that form non-vanishing cubic terms project non-trivially onto both quark/lepton doublets and singlets, as well as Higgs and Higgs conjugate fields under the action of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group.

We conclude that non-vanishing Yukawa terms proportional to QHu, $Q\bar{H}d$, $LH\nu$ and $L\bar{H}e$ appear in the low energy superpotential of a minimal heterotic standard model. However, their structure is constrained by the above selection rules. The exact texture of the Yukawa interactions and its implications for the quark/lepton mass matrix are presented in Section 5. We show that, in a suitable basis, one out of the three quark/lepton families is, prior to higher order and non-perturbative corrections, massless. The remaining two generations have masses of the order of the electroweak symmetry breaking scale.

2 Preliminaries

2.1 Heterotic String on a Calabi-Yau Manifold

The observable sector of an $E_8 \times E_8$ heterotic standard model vacuum consists of a stable, holomorphic vector bundle, V, over a Calabi-Yau threefold, X. In particular, we are interested in an $SU(4)$ instanton, breaking the low energy gauge group down to its commutant

$$E_8 \xrightarrow{SU(4)} Spin(10) \ . \tag{1}$$

Additionally, we want $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines W. The $Spin(10)$ group is then spontaneously broken by the holonomy group of W to

$$Spin(10) \xrightarrow{\mathbb{Z}_3 \times \mathbb{Z}_3} SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \ . \tag{2}$$

In this way, the standard model gauge group emerges in the low energy effective theory multiplied by an additional $U(1)$ gauge group whose charges correspond to $B - L$ quantum numbers.

For W to exist, the Calabi-Yau manifold X must have fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_3$. The physical properties of this vacuum are most easily deduced not from X and V but, rather, from two closely related entities, which we denote by $\widetilde{X}$ and $\widetilde{V}$ respectively. $\widetilde{X}$

is a simply-connected Calabi-Yau threefold which admits a freely acting $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action such that

$$X = \widetilde{X} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big). \tag{3}$$

That is, $\widetilde{X}$ is the universal covering space of X. Similarly, $\widetilde{V}$ is a stable, holomorphic vector bundle over $\widetilde{X}$ with structure group $SU(4)$ which is equivariant under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then,

$$V = \widetilde{V} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big). \tag{4}$$

The covering space $\widetilde{X}$ for a heterotic standard model was discussed in detail in [30]. Here, it suffices to recall that $\widetilde{X}$ is a fiber product

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 \tag{5}$$

of two rational elliptic (dP_9) surfaces B_1 and B_2 with $\mathbb{Z}_3 \times \mathbb{Z}_3$ action. Thus, $\widetilde{X}$ is elliptically fibered over both surfaces with the projections

$$\pi_1 : \widetilde{X} \to B_1 \,, \quad \pi_2 : \widetilde{X} \to B_2 \,. \tag{6}$$

The surfaces B_1 and B_2 are themselves elliptically fibered over $\mathbb{P}^1$ with maps

$$\beta_1 : B_1 \to \mathbb{P}^1 \,, \quad \beta_2 : B_2 \to \mathbb{P}^1 \,. \tag{7}$$

Together, these projections yield the commutative diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B_1 \arrow[dr, "\beta_1"'] & & B_2 \arrow[dl, "\beta_2"] \\ & \mathbb{P}^1 \,. & \end{tikzcd} \tag{8}$$

The invariant homology ring of each special dP_9 surface is generated by two $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant curve classes f and t. Using the projections in eq. (6), these can be pulled back to divisor classes

$$\tau_1 = \pi_1^{-1}(t_1) \,, \quad \tau_2 = \pi_2^{-1}(t_2) \,, \quad \phi = \pi_1^{-1}(f_1) = \pi_2^{-1}(f_2) \tag{9}$$

on $\widetilde{X}$. These three classes generate the even invariant homology ring of $\widetilde{X}$. In particular,

$$\operatorname{span}\{\tau_1, \tau_2, \phi\} = H^2\Big(\widetilde{X}, \mathbb{C}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \tag{10}$$

is the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant part of the Kähler moduli space.

2.2 The Gauge Bundle

The crucial ingredient in a heterotic standard model is the choice of the observable sector vector bundle $\widetilde{V}$. These bundles are constructed using a generalization of the method of *bundle extensions* [27, 29]. Specifically, $\widetilde{V}$ is the extension

$$0 \longrightarrow V_1 \longrightarrow \widetilde{V} \longrightarrow V_2 \longrightarrow 0 \tag{11}$$

of two rank two bundles V_1 and V_2 on $\widetilde{X}$. The solution for V_1 and V_2 leading to the minimal heterotic standard model is as follows. Define

$$V_1 = \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \otimes \pi_1^*(W_1)\,, \quad V_2 = \mathcal{O}_{\widetilde{X}}(\tau_1 - \tau_2) \otimes \pi_2^*(W_2)\,, \tag{12}$$

where $\mathcal{O}_{\widetilde{X}}(\mp\tau_1 \pm \tau_2)$ are line bundles on $\widetilde{X}$ and the rank 2 bundles W_1, W_2 are constructed via an equivariant version of the Serre construction as

$$0 \longrightarrow \chi_1 \mathcal{O}_{B_1}(-f_1) \longrightarrow W_1 \longrightarrow \chi_1^2 \mathcal{O}_{B_1}(f_1) \otimes I_3^{B_1} \longrightarrow 0 \tag{13}$$

and

$$0 \longrightarrow \chi_2^2 \mathcal{O}_{B_2}(-f_2) \longrightarrow W_2 \longrightarrow \chi_2 \mathcal{O}_{B_2}(f_2) \otimes I_6^{B_2} \longrightarrow 0\,, \tag{14}$$

where $I_3^{B_1}$ and $I_6^{B_2}$ denote the ideal sheaf[1] of 3 and 6 points in B_1 and B_2 respectively. The characters χ_1 and χ_2 are third roots of unity which generate the first and second factors of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The observable sector equivariant bundle $\widetilde{V}$ is then an invariant element of the space of extensions defined in eq. (11). The vector bundle $\widetilde{V}$ so-constructed is slope-stable [7].

Let R be any representation of $Spin(10)$ and $U(\widetilde{V})_R$ the associated tensor product bundle of $\widetilde{V}$. Then, each sheaf cohomology space $H^*\big(\widetilde{X}, U(\widetilde{V})_R\big)$ carries a specific representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Similarly, the Wilson line W manifests itself as a $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on each representation R of $Spin(10)$. As discussed in detail in [4], the low-energy particle spectrum is given by

$$\begin{aligned} \ker\,(\not{\partial}_{\widetilde{V}}) = & \Big(H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big) \otimes \mathbf{45}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus \Big(H^1\big(\widetilde{X}, \widetilde{V}^\vee\big) \otimes \overline{\mathbf{16}}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \\ \oplus & \Big(H^1\big(\widetilde{X}, \widetilde{V}\big) \otimes \mathbf{16}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus \Big(H^1\big(\widetilde{X}, \wedge^2 \widetilde{V}\big) \otimes \mathbf{10}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus \Big(H^1\big(\widetilde{X}, \mathrm{ad}(\widetilde{V})\big) \otimes \mathbf{1}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}\,, \end{aligned} \tag{15}$$

where the superscript indicates the invariant subspace under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The invariant cohomology space $\big(H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \otimes \mathbf{45}\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ corresponds to gauge superfields in the low-energy spectrum carrying the adjoint representation of the gauge group. The matter cohomology spaces

$$\Big(H^1(\widetilde{X}, \widetilde{V}^\vee) \otimes \overline{\mathbf{16}}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}, \quad \Big(H^1(\widetilde{X}, \widetilde{V}) \otimes \mathbf{16}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}, \quad \Big(H^1(\widetilde{X}, \wedge^2 \widetilde{V}) \otimes \mathbf{10}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \tag{16}$$

[1]The analytic functions vanishing at the respective points.

were all explicitly computed in [4]. One finds that $H^1\big(\widetilde{X}, \widetilde{V}^\vee\big) = 0$ and, hence, there are no vector-like pairs of quark/lepton families. The space $\big(H^1(\widetilde{X}, \widetilde{V}) \otimes \mathbf{16}\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ consists of three chiral families of quarks/leptons, each family with a right-handed neutrino [47], and transforming as

$$Q = \big(\mathbf{3}, \mathbf{2}, 1, 1\big), \quad u = \big(\overline{\mathbf{3}}, \mathbf{1}, -4, -1\big), \quad d = \big(\overline{\mathbf{3}}, \mathbf{1}, 2, -1\big) \tag{17}$$

and

$$L = \big(\mathbf{1}, \mathbf{2}, -3, -3\big), \quad e = \big(\mathbf{1}, \mathbf{1}, 6, 3\big), \quad \nu = \big(\mathbf{1}, \mathbf{1}, 0, 3\big) \tag{18}$$

under $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$. We have displayed the quantum numbers $3Y$ and $3(B-L)$ for convenience. The cohomology space $\big(H^1(\widetilde{X}, \wedge^2 \widetilde{V}) \otimes \mathbf{10}\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ is spanned by one vector-like pair of Higgs–Higgs conjugate superfields

$$H = \big(\mathbf{1}, \mathbf{2}, 3, 0\big), \quad \bar{H} = \big(\mathbf{1}, \overline{\mathbf{2}}, -3, 0\big). \tag{19}$$

That is, the matter spectrum is precisely that of the MSSM. The remaining cohomology space, $\big(H^1(\widetilde{X}, \mathrm{ad}(\widetilde{V})) \otimes \mathbf{1}\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$, was computed using the formalism introduced in [43] and corresponds to 13 vector bundle moduli.

2.3 Cubic Terms in the Superpotential

In this paper, we will focus on computing Yukawa terms. It follows from eq. (15) that the 4-dimensional Higgs and quark/lepton fields correspond to certain $\bar{\partial}$-closed $(0,1)$-forms on $\widetilde{X}$ with values in the vector bundle $\wedge^2 \widetilde{V}$ and $\widetilde{V}$ respectively. Since both H and $\bar{H}$ arise from the same cohomology space, we will denote either of these 1-forms simply as Ψ^H. For the same reason, we will schematically represent any quark/lepton doublet by $\Psi^{(2)}$ and any singlet 1-form by $\Psi^{(1)}$, in any family. They can be written as

$$\Psi^H = \psi^H_{\bar{\iota}[ab]}\, \mathrm{d}\bar{z}^{\bar{\iota}}, \quad \Psi^{(1)} = \psi^{(1)}_{\bar{\iota}a}\, \mathrm{d}\bar{z}^{\bar{\iota}}, \quad \Psi^{(2)} = \psi^{(2)}_{\bar{\iota}b}\, \mathrm{d}\bar{z}^{\bar{\iota}}, \tag{20}$$

where a, b are valued in the $SU(4)$ bundle $\widetilde{V}$ and $\{z^\iota, \bar{z}^{\bar{\iota}}\}$ are coordinates on the Calabi-Yau threefold $\widetilde{X}$. Doing the dimensional reduction of the 10-dimensional Lagrangian yields cubic terms in the superpotential of the 4-dimensional effective action. It turns out [13] that the coefficients of the cubic couplings are simply the various allowed ways to obtain a number out of the forms Ψ^H, $\Psi^{(1)}$, $\Psi^{(2)}$. That is

$$W = \cdots + \lambda_u Q H u + \lambda_d Q \bar{H} d + \lambda_\nu L H \nu + \lambda_e L \bar{H} e \tag{21}$$

with the coefficients λ determined by

$$\begin{aligned} \lambda &= \int_{\widetilde{X}} \Omega \wedge \mathrm{Tr}\Big[\Psi^{(2)} \wedge \Psi^H \wedge \Psi^{(1)}\Big] = \\ &= \int_{\widetilde{X}} \Omega \wedge \Big(\epsilon^{abcd} \psi^{(2)}_{\bar{\iota}a}\, \psi^H_{\bar{\kappa}[bc]}\, \psi^{(1)}_{\bar{\epsilon}d}\Big) \mathrm{d}\bar{z}^{\bar{\iota}} \wedge \mathrm{d}\bar{z}^{\bar{\kappa}} \wedge \mathrm{d}\bar{z}^{\bar{\epsilon}} \end{aligned} \tag{22}$$

and Ω is the holomorphic $(3,0)$-form. Mathematically, we are using the wedge product together with a contraction of the vector bundle indices (that is, the determinant $\wedge^4 \widetilde{V} = \mathcal{O}_{\widetilde{X}}$) to obtain a product

$$H^1\Big(\widetilde{X}, \widetilde{V}\Big) \otimes H^1\Big(\widetilde{X}, \wedge^2 \widetilde{V}\Big) \otimes H^1\Big(\widetilde{X}, \widetilde{V}\Big) \longrightarrow \\ \longrightarrow H^3\Big(\widetilde{X}, \widetilde{V} \otimes \wedge^2 \widetilde{V} \otimes \widetilde{V}\Big) \longrightarrow H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) \quad (23)$$

plus the fact that on the Calabi-Yau manifold $\widetilde{X}$

$$H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) = H^3\Big(\widetilde{X}, K_{\widetilde{X}}\Big) = H^{3,3}_{\bar{\partial}}\Big(\widetilde{X}\Big) = H^6\Big(\widetilde{X}\Big) \quad (24)$$

can be integrated over. If one were to use the heterotic string with the "standard embedding", then the above product would simplify further to the intersection of certain cycles in the Calabi-Yau threefold [48, 49]. However, in our case there is no such description.

Hence, to compute Yukawa terms, we must first analyze the cohomology groups

$$H^1\Big(\widetilde{X}, \widetilde{V}\Big),\ H^1\Big(\widetilde{X}, \wedge^2 \widetilde{V}\Big),\ H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) \quad (25)$$

and the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on these spaces. We then have to evaluate the product in eq. (23). As we will see in the following sections, the two independent elliptic fibrations of $\widetilde{X}$ will force some, but not all, products to vanish.

3 The First Elliptic Fibration

3.1 The Leray Spectral Sequence

As discussed in detail in [4], the cohomology spaces on $\widetilde{X}$ are obtained by using two Leray spectral sequences. In this section, we consider the first of these sequences corresponding to the projection

$$\widetilde{X} \xrightarrow{\pi_2} B_2. \quad (26)$$

For any sheaf $\mathcal{F}$ on $\widetilde{X}$, the Leray spectral sequence tells us that[2]

$$H^i\Big(\widetilde{X}, \mathcal{F}\Big) = \bigoplus_{p,q}^{p+q=i} H^p\Big(B_2, R^q \pi_{2*} \mathcal{F}\Big), \quad (27)$$

[2]In all the spectral sequences we are considering in this paper, higher differentials vanish trivially. Hence, the E_2 and E_∞ tableaux are equal and we will not distinguish them in the following. Furthermore, there are no extension ambiguities for $\mathbb{C}$-vector spaces.

where the only non-vanishing entries are for $p = 0, 1, 2$ (since $\dim_{\mathbb{C}}(B_2) = 2$) and $q = 0, 1$ (since the fiber of $\widetilde{X}$ is an elliptic curve, therefore of complex dimension one). Note that the cohomologies $H^p(B_2, R^q\pi_{2*}\mathcal{F})$ fill out the 2×3 tableau[3]

$$\begin{array}{r|c|c|c|l}
\cline{2-4}
{\scriptstyle q=1} & H^0\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^1\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^2\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & \\
\cline{2-4}
{\scriptstyle q=0} & H^0\big(B_2, \pi_{2*}\mathcal{F}\big) & H^1\big(B_2, \pi_{2*}\mathcal{F}\big) & H^2\big(B_2, \pi_{2*}\mathcal{F}\big) & \Rightarrow H^{p+q}\Big(\widetilde{X}, \mathcal{F}\Big), \\
\cline{2-4}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{\scriptstyle p=0} & \multicolumn{1}{c}{\scriptstyle p=1} & \multicolumn{1}{c}{\scriptstyle p=2} &
\end{array} \tag{28}$$

where "$\Rightarrow H^{p+q}\big(\widetilde{X}, \mathcal{F}\big)$" reminds us of which cohomology group the tableau is computing. Such tableaux are very useful in keeping track of the elements of Leray spectral sequences. As is clear from eq. (27), the sum over the diagonals yields the desired cohomology of $\mathcal{F}$. In the following, it will be very helpful to define

$$H^p\Big(B_2, R^q\pi_{2*}\mathcal{F}\Big) \equiv \big(p, q\big|\mathcal{F}\big). \tag{29}$$

Using this abbreviation, the tableau eq. (28) reads

$$\begin{array}{r|c|c|c|l}
\cline{2-4}
{\scriptstyle q=1} & \big(0,1\big|\mathcal{F}\big) & \big(1,1\big|\mathcal{F}\big) & \big(2,1\big|\mathcal{F}\big) & \\
\cline{2-4}
{\scriptstyle q=0} & \big(0,0\big|\mathcal{F}\big) & \big(1,0\big|\mathcal{F}\big) & \big(2,0\big|\mathcal{F}\big) & \Rightarrow H^{p+q}\Big(\widetilde{X}, \mathcal{F}\Big). \\
\cline{2-4}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{\scriptstyle p=0} & \multicolumn{1}{c}{\scriptstyle p=1} & \multicolumn{1}{c}{\scriptstyle p=2} &
\end{array} \tag{30}$$

3.2 Degrees and Products

On the level of differential forms, we can understand the Leray spectral sequence as decomposing differential forms into the number p of legs in the direction of the base and the number q of legs in the fiber direction. Obviously, this extra grading is preserved under the wedge-product of the differential forms. Hence, any product

$$H^i\Big(\widetilde{X}, \mathcal{F}_1\Big) \otimes H^j\Big(\widetilde{X}, \mathcal{F}_2\Big) \longrightarrow H^{i+j}\Big(\widetilde{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\Big) \tag{31}$$

not only has to end up in overall degree $i+j$, but also has to preserve the (p, q)-grading. That is,

$$\begin{array}{ccccc}
\big(p_1, q_1\big|\mathcal{F}_1\big) & \otimes & \big(p_2, q_2\big|\mathcal{F}_2\big) & \longrightarrow & \big(p_1+p_2, q_1+q_2\big|\mathcal{F}_1 \otimes \mathcal{F}_2\big) \\
\cap & & \cap & & \cap \\
H^{p_1+q_1}\Big(\widetilde{X}, \mathcal{F}_1\Big) & \otimes & H^{p_2+q_2}\Big(\widetilde{X}, \mathcal{F}_2\Big) & \longrightarrow & H^{p_1+p_2+q_1+q_2}\Big(\widetilde{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\Big).
\end{array} \tag{32}$$

This is all we are going to need in the following, but we would like to mention the following caveat. Although it does not happen here, sometimes the push-down is not a

[3] Recall that the zero-th derived push-down is just the ordinary push-down, $R^0\pi_{2*} = \pi_{2*}$.

vector bundle, but a (non-locally free) sheaf. Then the identification with bundle-valued differential forms is not possible. The way around this is well-known; one has to replace the coherent sheaf by a complex of vector bundles. Now one can again think in terms of differential forms, but at the cost of working in the derived category. What can and does happen in general is the appearance of derived tensor products. That is, the tensor product of complexes may no longer be quasi-isomorphic to a complex with only one entry. The effect is that the product ends up in

$$\left(p_1, q_1 \middle| \mathcal{F}_1\right) \otimes \left(p_2, q_2 \middle| \mathcal{F}_2\right) \longrightarrow \bigoplus_{n=0}^{\min(\mathrm{hd}(\mathcal{F}_1),\mathrm{hd}(\mathcal{F}_2))} \left(p_1 + p_2 + n, q_1 + q_2 - n \middle| \mathcal{F}_1 \otimes \mathcal{F}_2\right), \quad (33)$$

where $\mathrm{hd}\left(\mathcal{F}_i\right)+1$ is the length of the shortest locally free resolution of $\mathcal{F}_i$. In all products that occur in this paper $\mathrm{hd}(\mathcal{F}) = 0$ and, hence, eq. (33) simplifies to eq. (32).

3.3 The First Leray Decomposition of the Volume Form

Let us first discuss the (p, q) Leray tableau for the sheaf $\mathcal{F} = \mathcal{O}_{\widetilde{X}}$, which is the last term in eq. (25). Since this is the trivial line bundle, it immediately follows that

$$\begin{array}{c|c|c|c|}
q{=}1 & 0 & 0 & \mathbf{1} \\
\hline
q{=}0 & \mathbf{1} & 0 & 0 \\
\hline
 & p{=}0 & p{=}1 & p{=}2
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big). \quad (34)$$

From eqns. (27) and (34) we see that

$$H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) = \left(2, 1 \middle| \mathcal{O}_{\widetilde{X}}\right) = \mathbf{1}, \quad (35)$$

where the $\mathbf{1}$ indicates that $H^3(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is a one-dimensional space carrying the trivial action of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

3.4 The First Leray Decomposition of Higgs Fields

Now consider the (p, q) Leray tableau for the sheaf $\mathcal{F} = \wedge^2 \widetilde{V}$, which is the second term in eq. (25). This can be explicitly computed and is given by

$$\begin{array}{c|c|c|c|}
q{=}1 & 0 & \rho_4 & 0 \\
\hline
q{=}0 & 0 & \rho_4 & 0 \\
\hline
 & p{=}0 & p{=}1 & p{=}2
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, \wedge^2 \widetilde{V}\Big), \quad (36)$$

where ρ_4 is the four-dimensional representation

$$\rho_4 = \chi_2 \oplus \chi_2^2 \oplus \chi_1 \chi_2^2 \oplus \chi_1^2 \chi_2 \quad (37)$$

of $\mathbb{Z}_3 \times \mathbb{Z}_3$. In general, it follows from eq. (27) that $H^1(\widetilde{X}, \wedge^2 \widetilde{V})$ is the sum of the two subspaces $\left(0,1 \middle| \wedge^2 \widetilde{V}\right) \oplus \left(1,0 \middle| \wedge^2 \widetilde{V}\right)$. However, we see from the Leray tableau eq. (36) that the $\left(0,1 \middle| \wedge^2 \widetilde{V}\right)$ space vanishes. Hence,

$$H^1\left(\widetilde{X}, \wedge^2 \widetilde{V}\right) = \left(1,0 \middle| \wedge^2 \widetilde{V}\right) = \rho_4. \tag{38}$$

3.5 The First Leray Decomposition of the Quark/Lepton Fields

Now consider the (p,q) Leray tableau for the sheaf $\mathcal{F} = \widetilde{V}$, which is the first term in eq. (25). This can be explicitly computed and is given by

$$\begin{array}{r|c|c|c|l} \cline{2-4} {\scriptstyle q=1} & RG & 0 & 0 & \\ \cline{2-4} {\scriptstyle q=0} & 0 & RG^{\oplus 2} & 0 & \Rightarrow H^{p+q}\left(\widetilde{X}, \widetilde{V}\right), \\ \cline{2-4} \multicolumn{1}{r}{} & \multicolumn{1}{c}{\scriptstyle p=0} & \multicolumn{1}{c}{\scriptstyle p=1} & \multicolumn{1}{c}{\scriptstyle p=2} & \end{array} \tag{39}$$

where RG is the regular representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$ given by

$$RG = 1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1^2 \oplus \chi_2^2 \oplus \chi_1\chi_2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2 \oplus \chi_1^2\chi_2^2. \tag{40}$$

It follows from eq. (27) that $H^1(\widetilde{X}, \widetilde{V})$ is the sum of the two subspaces

$$H^1\left(\widetilde{X}, \widetilde{V}\right) = \left(0,1 \middle| \widetilde{V}\right) \oplus \left(1,0 \middle| \widetilde{V}\right). \tag{41}$$

Furthermore, eq. (39) tells us that

$$\left(0,1 \middle| \widetilde{V}\right) = RG, \quad \left(1,0 \middle| \widetilde{V}\right) = RG^{\oplus 2}. \tag{42}$$

Technically, the structure of eq. (41) is associated with the fact that the cohomology $H^*\left(\widetilde{X}, \widetilde{V}\right)$ decomposes into $H^*\left(\widetilde{X}, V_1\right) \oplus H^*\left(\widetilde{X}, V_2\right)$. It turns out that the two subspaces in eq. (41) arise as

$$RG = H^1\left(\widetilde{X}, V_1\right), \quad RG^{\oplus 2} = H^1\left(\widetilde{X}, V_2\right) \tag{43}$$

respectively.

3.6 The (p,q) Selection Rule

Having computed the decompositions of $H^3(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, $H^1(\widetilde{X}, \wedge^2 \widetilde{V})$ and $H^1(\widetilde{X}, \widetilde{V})$ into their (p,q) Leray subspaces, we can now analyze the (p,q) components of the triple product

$$H^1\left(\widetilde{X}, \widetilde{V}\right) \otimes H^1\left(\widetilde{X}, \wedge^2 \widetilde{V}\right) \otimes H^1\left(\widetilde{X}, \widetilde{V}\right) \longrightarrow H^3\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\right) \tag{44}$$

given in eq. (23). Inserting eqns. (38) and (41), we see that

$$
\begin{gathered}
H^1\Big(\widetilde{X},\widetilde{V}\Big)\otimes H^1\Big(\widetilde{X},\wedge^2\widetilde{V}\Big)\otimes H^1\Big(\widetilde{X},\widetilde{V}\Big)= \\
\Big(\big(0,1\big|\widetilde{V}\big)\oplus\big(1,0\big|\widetilde{V}\big)\Big)\otimes\big(1,0\big|\wedge^2\widetilde{V}\big)\otimes\Big(\big(0,1\big|\widetilde{V}\big)\oplus\big(1,0\big|\widetilde{V}\big)\Big)= \\
\underbrace{\Big(\big(0,1\big|\widetilde{V}\big)\otimes\big(1,0\big|\wedge^2\widetilde{V}\big)\otimes\big(1,0\big|\widetilde{V}\big)\Big)^{\oplus 2}}_{\text{total }(p,q)\text{ degree }=(2,1)}\oplus\underbrace{\Big(\big(1,0\big|\widetilde{V}\big)\otimes\big(1,0\big|\wedge^2\widetilde{V}\big)\otimes\big(1,0\big|\widetilde{V}\big)\Big)}_{\text{total }(p,q)\text{ degree }=(3,0)}\oplus\underbrace{\Big(\big(0,1\big|\widetilde{V}\big)\otimes\big(0,1\big|\wedge^2\widetilde{V}\big)\otimes\big(0,1\big|\widetilde{V}\big)\Big)}_{\text{total }(p,q)\text{ degree }=(0,3)}
\end{gathered}
\tag{45}
$$

Because of the (p,q) degree, we see from eq. (35) that only the first term can have a non-zero product in

$$H^3\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}\Big)=\big(2,1\big|\mathcal{O}_{\widetilde{X}}\big). \tag{46}$$

It follows that the first quark/lepton family, which arises from

$$\big(0,1\big|\widetilde{V}\big)=RG, \tag{47}$$

will form non-vanishing Yukawa terms with the second and third quark/lepton families coming from

$$\big(1,0\big|\widetilde{V}\big)=RG^{\oplus 2}. \tag{48}$$

All other Yukawa couplings must vanish. We refer to this as the (p,q) *Leray degree selection rule*. We conclude that the only non-zero product in eq. (44) is of the form

$$\big(0,1\big|\widetilde{V}\big)\otimes\big(1,0\big|\wedge^2\widetilde{V}\big)\otimes\big(1,0\big|\widetilde{V}\big)\longrightarrow\big(2,1\big|\mathcal{O}_{\widetilde{X}}\big). \tag{49}$$

Roughly what happens is the following. The holomorphic $(3,0)$-form Ω has two legs in the base and one leg in the fiber direction. According to eq. (38), both 1-forms Ψ^H corresponding to Higgs and Higgs conjugate have their one leg in the base direction. Therefore, the wedge product in eq. (22) can only be non-zero if one quark/lepton 1-form Ψ has its leg in the base direction and the other quark/lepton 1-form Ψ has its leg in the fiber direction.

We conclude that due to a selection rule for the (p,q) Leray degree, the Yukawa terms in the effective low energy theory can involve only a coupling of the first quark/lepton family to the second and third. All other Yukawa couplings must vanish.

4 The Second Elliptic Fibration

4.1 The Second Leray Spectral Sequence

So far, we only made use of the fact that our Calabi-Yau manifold is an elliptic fibration over the base B_2. But the dP_9 surface B_2 is itself elliptically fibered over $\mathbb{P}^1$. Consequently, there is yet another selection rule coming from the second elliptic fibration.

Therefore, we now consider the second Leray spectral sequence corresponding to the projection

$$B_2 \xrightarrow{\beta_2} \mathbb{P}^1. \tag{50}$$

For any sheaf $\widehat{\mathcal{F}}$ on B_2, the Leray sequence now starts with a 2×2 Leray tableau

$$\begin{array}{r|c|c|}
\hline
t{=}1 & H^0\big(\mathbb{P}^1, R^1\beta_{2*}\widehat{\mathcal{F}}\big) & H^1\big(\mathbb{P}^1, R^1\beta_{2*}\widehat{\mathcal{F}}\big) \\
\hline
t{=}0 & H^0\big(\mathbb{P}^1, \beta_{2*}\widehat{\mathcal{F}}\big) & H^1\big(\mathbb{P}^1, \beta_{2*}\widehat{\mathcal{F}}\big) \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1}
\end{array} \Rightarrow H^{s+t}\Big(B_2, \widehat{\mathcal{F}}\Big). \tag{51}$$

Again, the sum over the diagonals yields the desired cohomology of $\widehat{\mathcal{F}}$. Note that to evaluate the product eq. (49), we need the $[s,t]$ Leray tableaux for

$$\widehat{\mathcal{F}} = R^1\pi_{2*}\big(\widetilde{V}\big),\ \pi_{2*}\big(\widetilde{V}\big),\ \pi_{2*}\big(\wedge^2 \widetilde{V}\big),\ R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big). \tag{52}$$

In the following, it will be useful to define

$$H^s\bigg(\mathbb{P}^1, R^t\beta_{2*}\Big(R^q\pi_{2*}\big(\mathcal{F}\big)\Big)\bigg) \equiv \big[s,t\big|q,\mathcal{F}\big]. \tag{53}$$

One can think of $\big[s,t\big|q,\mathcal{F}\big]$ as the subspace of $H^*\big(\widetilde{X},\mathcal{F}\big)$ that can be written as forms with q legs in the π_2-fiber direction, t legs in the β_2-fiber direction, and s legs in the base $\mathbb{P}^1$ direction.

4.2 The Second Leray Decomposition of the Volume Form

Let us first discuss the $[s,t]$ Leray tableau for $\widehat{\mathcal{F}} = R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big) = K_{B_2}$, the canonical line bundle. It follows immediately that

$$\begin{array}{r|c|c|}
\hline
t{=}1 & 0 & \mathbf{1} \\
\hline
t{=}0 & 0 & 0 \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1}
\end{array} \Rightarrow H^{s+t}\Big(B_2, R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big)\Big). \tag{54}$$

In our notation, this means that

$$H^2\Big(B_2, R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big)\Big) = \big[1,1\big|1,\mathcal{O}_{\widetilde{X}}\big] \tag{55}$$

has pure $[s,t] = [1,1]$ degree. To summarize, we see that

$$H^3\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}\Big) = \big(2,1\big|\mathcal{O}_{\widetilde{X}}\big) = \big[1,1\big|1,\mathcal{O}_{\widetilde{X}}\big] = \mathbf{1}. \tag{56}$$

4.3 The Second Leray Decomposition of Higgs Fields

Now consider the $[s,t]$ Leray tableau for the sheaf $\widehat{\mathcal{F}} = \pi_{2*}\big(\wedge^2 \widetilde{V}\big)$. This can be explicitly computed and is given by

$$\begin{array}{r|c|c|} \hline t{=}1 & \chi_1\chi_2^2 & 0 \\ \hline t{=}0 & 0 & \chi_2 \oplus \chi_2^2 \oplus \chi_1^2\chi_2 \\ \hline \multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1} \end{array} \Rightarrow H^{s+t}\Big(B_2, \pi_{2*}\big(\wedge^2 \widetilde{V}\big)\Big). \tag{57}$$

This means that the 4 copies of the **10** of $Spin(10)$ given in eq. (38) split as

$$H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) = \big(1,0\big|\wedge^2\widetilde{V}\big) = \big[0,1\big|0,\wedge^2\widetilde{V}\big] \oplus \big[1,0\big|0,\wedge^2\widetilde{V}\big], \tag{58}$$

where

$$\begin{aligned} \big[0,1\big|0,\wedge^2\widetilde{V}\big] &= \chi_1\chi_2^2 \\ \big[1,0\big|0,\wedge^2\widetilde{V}\big] &= \chi_2 \oplus \chi_2^2 \oplus \chi_1^2\chi_2. \end{aligned} \tag{59}$$

Note that

$$\big[0,1\big|0,\wedge^2\widetilde{V}\big] \oplus \big[1,0\big|0,\wedge^2\widetilde{V}\big] = \rho_4 \tag{60}$$

in eq. (37), as it must.

4.4 The Second Leray Decomposition of the Quark/Lepton Fields

Finally, let us consider the $[s,t]$ Leray tableau for the quark/lepton fields. We have already seen that, due to the (p,q) selection rule, both the first quark/lepton family arising from

$$\big(0,1\big|\widetilde{V}\big) = RG \tag{61}$$

and the second and third quark/lepton families coming from

$$\big(1,0\big|\widetilde{V}\big) = RG^{\oplus 2} \tag{62}$$

must occur in non-vanishing Yukawa interactions. Therefore, we are only interested in the $[s,t]$ decomposition of each of these subspaces. The $\big(0,1\big|\widetilde{V}\big)$ subspace is associated with the degree 0 cohomology of the sheaf $R^1\pi_{2*}\big(\widetilde{V}\big)$. The corresponding Leray tableau is given by

$$\begin{array}{r|c|c|} \hline t{=}1 & 0 & 0 \\ \hline t{=}0 & RG & 0 \\ \hline \multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1} \end{array} \Rightarrow H^{s+t}\Big(B_2, R^1\pi_{2*}\big(\widetilde{V}\big)\Big). \tag{63}$$

It follows that the first family of quarks/leptons has $[s,t]$ degree $[0,0]$,

$$\big(0,1\big|\widetilde{V}\big) = \big[0,0\big|1,\widetilde{V}\big] = RG. \tag{64}$$

The $\left(1,0\middle|\widetilde{V}\right)$ subspace is associated with the degree 1 cohomology of the sheaf $\pi_{2*}\big(\widetilde{V}\big)$. The corresponding Leray tableau is given by

$$\begin{array}{c|c|c|}
\cline{2-3}
t{=}1 & RG^{\oplus 2} & 0 \\
\cline{2-3}
t{=}0 & 0 & 0 \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1}
\end{array} \Rightarrow H^{s+t}\Big(B_2, \pi_{2*}\big(\widetilde{V}\big)\Big). \tag{65}$$

It follows that the second and third families of quarks/leptons has $[s,t]$ degree $[0,1]$,

$$\left(1,0\middle|\widetilde{V}\right) = \left[0,1\middle|1,\widetilde{V}\right] = RG^{\oplus 2}. \tag{66}$$

4.5 The [s,t] Selection Rule

Having computed the decompositions of the relevant cohomology spaces into their $[s,t]$ Leray subspaces, we can now calculate the triple product eq. (23). The (p,q) selection rule dictates that the only non-zero product is of the form eq. (49). Now split each term in this product into its $[s,t]$ subspaces, as given in eqns. (56), (59), and (64) respectively. The result is

$$\left[0,0\middle|1,\widetilde{V}\right] \otimes \Big(\left[0,1\middle|0,\wedge^2\widetilde{V}\right] \oplus \left[1,0\middle|0,\wedge^2\widetilde{V}\right]\Big) \otimes \left[0,1\middle|1,\widetilde{V}\right] \longrightarrow \left[1,1\middle|1,\mathcal{O}_{\widetilde{X}}\right]. \tag{67}$$

Clearly, this triple product vanishes by degree unless we choose the $\left[1,0\middle|0,\wedge^2\widetilde{V}\right]$ from the $\left(1,0\middle|\wedge^2\widetilde{V}\right)$ subspace. In this case, eq. (67) becomes

$$\left[0,0\middle|1,\widetilde{V}\right] \otimes \left[1,0\middle|0,\wedge^2\widetilde{V}\right] \otimes \left[0,1\middle|1,\widetilde{V}\right] \longrightarrow \left[1,1\middle|1,\mathcal{O}_{\widetilde{X}}\right], \tag{68}$$

which is consistent.

We conclude that there is, in addition to the (p,q) selection rule discussed above, a $[s,t]$ *Leray degree selection rule.* This rule continues to allow non-vanishing Yukawa couplings of the first quark/lepton family with the second and third quark/lepton families, but only through the

$$\left[1,0\middle|0,\wedge^2\widetilde{V}\right] = \chi_2 \oplus \chi_2^2 \oplus \chi_1^2\chi_2 \tag{69}$$

component of $\left(1,0\middle|\wedge^2\widetilde{V}\right)$ in eq. (58).

4.6 Wilson Lines

We have, in addition to the $SU(4)$ instanton, a non-vanishing Wilson line. Its effect is to break the $Spin(10)$ gauge group down to the desired $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$ gauge group. First, consider the **16** matter representations. We choose the Wilson line W so that its $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on each **16** is given by

$$\mathbf{16} = \Big[\chi_1\chi_2^2 Q \oplus \chi_2^2 e \oplus \chi_1^2\chi_2^2 u\Big] \oplus \Big[L \oplus \chi_1^2 d\Big] \oplus \chi_2\nu, \tag{70}$$

where the representations Q,u,d and L,ν,e were defined in eqns. (17) and (18), respectively. Recall from eqns. (41) and (42) that $H^1\big(\widetilde{X},\widetilde{V}\big) = RG \oplus RG^{\oplus 2}$. Tensoring any RG subspace of the cohomology space $H^1\big(\widetilde{X},\widetilde{V}\big)$ with a $\mathbf{16}$ using eqns. (40) and (70), we find that the invariant subspace under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action is

$$\Big(RG \otimes \mathbf{16}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \operatorname{span}\big\{Q, u, d, L, \nu, e\big\} \tag{71}$$

It follows that each RG subspace of $H^1\big(\widetilde{X},\widetilde{V}\big)$ projects to a complete quark/lepton family at low energy. This justifies our identification of the subspace RG with the first quark/lepton family and the subspace $RG^{\oplus 2}$ with the second and third quark/lepton families throughout the text.

Second, notice that each fundamental matter field in the $\mathbf{10}$ can be broken to a Higgs field, a color triplet, or projected out. In particular, we are going to choose the Wilson line W so that its $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on a $\mathbf{10}$ representation of $Spin(10)$ is given by

$$\mathbf{10} = \Big[\chi_2^2 H \oplus \chi_1^2\chi_2^2 C\Big] \oplus \Big[\chi_2 \bar{H} \oplus \chi_1\chi_2 \bar{C}\Big], \tag{72}$$

where H and $\bar{H}$ are defined in eq. (19) and

$$C = \big(\mathbf{3}, \mathbf{1}, -2, -2\big), \quad \bar{C} = \big(\overline{\mathbf{3}}, \mathbf{1}, 2, 2\big) \tag{73}$$

are the color triplet representations of $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$. Tensoring this with the cohomology space $H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big)$, we find the invariant subspace under the combined $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on the cohomology space and the Wilson line to be

$$\Big(H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big) \otimes \mathbf{10}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \operatorname{span}\big\{H, \bar{H}\big\}. \tag{74}$$

Hence, precisely one pair of Higgs–Higgs conjugate fields survives the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient. As required for any realistic model, all color triplets are projected out. The new information now is the (p,q) and $[s,t]$ degrees of the Higgs fields. Using the decompositions eqns. (38) and (58) of $H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big)$, we find

$$\begin{aligned}\Big(H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big) \otimes \mathbf{10}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} &= \Big(\big(1,0\big|\wedge^2\widetilde{V}\big) \otimes \mathbf{10}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \\ &= \underbrace{\Big(\big[0,1\big|0,\wedge^2\widetilde{V}\big] \otimes \mathbf{10}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}}_{=0} \oplus \underbrace{\Big(\big[1,0\big|0,\wedge^2\widetilde{V}\big] \otimes \mathbf{10}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}}_{=\operatorname{span}\{H,\bar{H}\}}.\end{aligned} \tag{75}$$

The dimensions and basis' of the two terms on the right side of this expression are determined by taking the tensor product of eqns. (59) and (72) and keeping the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant part. Note that the subspace forming the non-zero Yukawa couplings in eq. (68), namely $\big[1,0\big|0,\wedge^2\widetilde{V}\big]$, indeed projects to the Higgs–Higgs conjugate pair in the low energy theory.

5 Yukawa Couplings

To conclude, we analyzed cubic terms in the superpotential of the form

$$\lambda_{u,ij} Q_i H u_j, \quad \lambda_{d,ij} Q_i \bar{H} d_j, \quad \lambda_{\nu,ij} L_i H \nu_j, \quad \lambda_{e,ij} L_i \bar{H} e_j \tag{76}$$

where

- each coefficient λ is determined by an integral of the form of eq. (22),
- Q_i,L_i for $i = 1, 2, 3$ are the electroweak doublets of the three quarks/lepton families respectively,
- u_j,d_j,ν_j,e_j for $j = 1, 2, 3$ are the electroweak singlets of the three quark/lepton families respectively,
- H is the Higgs field, and
- $\bar{H}$ is the Higgs conjugate field.

We found that they are subject to two independent selection rules coming from the two independent torus fibrations. The first selection rule is that the total (p, q) degree is $(2, 1)$. Since the (p, q) degrees for the first quark/lepton family, the second and third quark/lepton families and the Higgs fields are $(0, 1)$, $(1, 0)$ and $(1, 0)$ respectively, it follows that the only non-vanishing λ coefficients are of the form

$$\lambda_{u,1j}, \lambda_{u,j1} \quad \lambda_{d,1j}, \lambda_{d,j1} \quad \lambda_{\nu,1j}, \lambda_{\nu,j1} \quad \lambda_{e,1j}, \lambda_{e,j1} \tag{77}$$

for $j = 2, 3$. That is, the only non-zero Yukawa terms couple the first family to the second and third families respectively. The second selection rule imposes independent constraints. It states that the total $[s, t]$ degree has to be $[1, 1]$. Of the two possible $[s, t]$ degrees associated with the Higgs fields, only the $[1, 0]$ subspace satisfies the $[s, t]$ selection rule. Happily, this is precisely the component that projects to a H-$\bar{H}$ pair at low energy. Hence, the conclusion in eq. (77) is unaltered.

Let us analyze, for example, the Yukawa contribution to the up-quark mass matrix. Assuming that H gets a non-vanishing vacuum expectation value $\langle H \rangle$ in its charge neutral component, this contribution can be written as

$$\begin{pmatrix} 0 & \lambda_{u,12}\langle H \rangle & \lambda_{u,13}\langle H \rangle \\ \lambda_{u,21}\langle H \rangle & 0 & 0 \\ \lambda_{u,31}\langle H \rangle & 0 & 0 \end{pmatrix} \tag{78}$$

Using independent non-singular transformations on the Q_i and u_i fields, one can find bases in which eq. (78) becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda\langle H \rangle & 0 \\ 0 & 0 & \lambda\langle H \rangle \end{pmatrix} \tag{79}$$

where λ is an arbitrary, but non-zero, real number. We conclude from the zero diagonal element that one up-quark is strictly massless[4]. Furthermore, the two non-zero diagonal elements imply that the second and third up-quarks will have non-vanishing masses of $O\big(\langle H\rangle\big)$. However, the exact value of their masses will depend on the explicit normalization of the kinetic energy terms in the low energy theory. These masses, therefore, are in general not degenerate. This analysis applies to the down-quarks and the up- and down-leptons as well. We conclude that, prior to higher order and non-perturbative corrections, one complete generation of quarks/leptons will be massless. The remaining two generations will have non-vanishing masses on the order of the electroweak symmetry breaking scale which are, generically, non-degenerate.

The coefficients λ have no interpretation as an intersection number and, therefore, no reason to be constant over the moduli space. In general, we expect them to depend on the moduli. Of course, to explicitly compute the quark/lepton masses one needs, in addition, the Kähler potential, which determines the correct normalization of the fields.

Acknowledgments

We are grateful to R. Donagi, B. Nelson, and T. Pantev for enlightening discussions. This research was supported in part by the Department of Physics and the Math/Physics Research Group at the University of Pennsylvania under cooperative research agreement DE-FG02-95ER40893 with the U. S. Department of Energy and an NSF Focused Research Grant DMS0139799 for "The Geometry of Superstrings." T. P. is partially supported by an NSF grant DMS 0403884. Y.-H. H. is also indebted to the FitzJames Fellowship of Merton College, Oxford.

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