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Algorithmic Algebraic Geometry and Flux Vacua

James Gray[1*], Yang-Hui He[2,3†], André Lukas[4‡],

[1]*Institut d'Astrophysique de Paris and APC, Université de Paris 7, 98 bis, Bd. Arago 75014, Paris, France*

[2]*Merton College, Oxford University, Oxford OX1 4JD, U.K.*

[3]*Mathematical Institute, Oxford University, 24-29 St. Giles', Oxford OX1 3LB, U.K.*

[4]*Rudolf Peierls Centre for Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, U.K.*

Abstract

We develop a new and efficient method to systematically analyse four dimensional effective supergravities which descend from flux compactifications. The issue of finding vacua of such systems, both supersymmetric and non-supersymmetric, is mapped into a problem in computational algebraic geometry. Using recent developments in computer algebra, the problem can then be rapidly dealt with in a completely algorithmic fashion. Two main results are (1) a procedure for calculating constraints which the flux parameters must satisfy in these models if any given type of vacuum is to exist; (2) a stepwise process for finding all of the isolated vacua of such systems and their physical properties. We illustrate our discussion with several concrete examples, some of which have eluded conventional methods so far.

1 Introduction

The issue of moduli stabilisation is one of the most pressing in string phenomenology today. Recent progress in this field has resulted in a variety of reasonably well-understood, completely stable vacua [1, 2, 3, 4, 5]. However, these vacua, for the most part, are not physical. Two of the greatest problems with these minima from a phenomenological standpoint are that they do not spontaneously break supersymmetry and that they give rise to an anti de

*gray@iap.fr

†hey@maths.ox.ac.uk

‡lukas@physics.ox.ac.uk

Sitter external space. Clearly, if we wish to use such vacua as a starting point for building a string theoretic description of our world this problem has to be addressed. In the literature, this issue is frequently resolved by employing some kind of "raising mechanism," for example, one based on the presence of anti-branes [4, 6], or on D-terms [7, 8]. In the context of a well-controlled supergravity descending from a string or M-theory model, there is, however, another option. In general, such theories rich in moduli will have vacua which exhibit spontaneously broken supersymmetry and which may be de Sitter - even in the absence of raising of any sort.

Finding such vacua is, however, a prohibitively difficult task using conventional methods. Generically, a large number of moduli fields are present in four dimensional effective descriptions of compactified theories. These describe such features of the internal space as its complex and Kähler structure or the form of some vector bundle, to name but a few. Therefore, one is confronted with potentials of supergravity theories as complicated functions in an overwhelming number of variables. Minimising such an expression can be beyond the reach of conventional techniques.

The purpose of this paper is to present a novel and efficient approach to the systematics of finding such flux vacua. In pedagogical detail we provide two basic tools which make the search for these extrema relatively easy. The first of these is a simple algorithmic process for generating constraints on the flux parameters in the superpotential which are necessary (and in some cases even sufficient) for the existence of vacua of any given type. The second tool we provide is a completely algorithmic way of finding all of the isolated vacua of a given system of interest - including non-supersymmetric vacua of the type described above. This tool is based upon a method for splitting up systems of polynomial equations into multiple systems of simpler such equalities. Thus, we start with a set of equations which describe *all* of the extrema of the potential and break these up into multiple sets of equations, where each of these new polynomial systems describes just *one* of the loci of extrema of the potential (say a single isolated vacuum). In the case of isolated vacua these new equations are so much simpler than the original expressions that it is found that one can solve them trivially. For example, to entice the reader, the following is one of the systems we discuss in later sections where we provide concrete examples of our methods:

$$\begin{aligned} K &= -4\log(-i(U-\bar{U})) - \log(-i(T_1-\bar{T}_1)(T_2-\bar{T}_2)(T_3-\bar{T}_3)), \\ W &= \frac{1}{\sqrt{8}}\left[4U(T_1+T_2+T_3) + 2T_2T_3 - T_1T_3 - T_1T_2 + 200\right] . \end{aligned} \tag{1}$$

This pair of Kähler potential and superpotential has been obtained in the literature by compactifying M-theory on a manifold of $SU(3)$ structure [9]. We call the associated scalar potential, as obtained from the usual supergravity formula, V. Solving for the vacua of this model directly by solving the equations $\partial V = 0$ is prohibitively difficult, at least as far as non-supersymmetric vacua are concerned. Instead, the method described in this paper starts by introducing a polynomial ideal $\langle \partial V \rangle$, obtained from the (polynomial) numerators of the partial derivatives of V. This ideal corresponds to the algebraic variety of extrema of V and can be decomposed into so-called primary ideals $P(i)$ by standard algorithms, so that $\langle \partial V \rangle = P_1 \cap \ldots \cap P_n$. Each of these primary ideals corresponds to an irreducible variety, or, in physical terms, a single branch of the vacuum space. Indeed, each $P(i)$ is much simpler than the original one and can be analysed explicitly in many cases. In particular, the zero-dimensional primary ideals which correspond to isolated extrema can be studied in detail using methods of real algebraic geometry. Applying primary decomposition to $\langle \partial V \rangle$ as obtained from the above model (subject to the additional, simplifying constraint $\mathrm{Re}(U) = 0$) leads to the following two zero-dimensional primary ideals:

$$\begin{aligned} &\{3x^2 = 100, t_1 = 2x, t_2 = x, t_3 = x, \tau_1 = 0, \tau_2 = 0, \tau_3 = 0, y = 0\} \ , \\ &\{9x^2 = 500, 5t_1 = 2x, t_2 = x, t_3 = x, \tau_1 = 0, \tau_2 = 0, \tau_3 = 0, y = 0\} \ . \end{aligned} \tag{2}$$

Here, we have defined $T_j = \tau_j + it_j$ for $j = 1, 2, 3$, and $U = y + ix$. Thus, by breaking the equations up in this manner using the techniques we will describe, we render the problem of finding isolated extrema of the potential, including its stabilised vacua, trivial. Even if cases were to exist where the simplification were not so drastic, this still would not constitute an obstacle for us. This is because we provide, in addition, practical algorithmic methods which can extract all of the properties of the vacua from these equations, without ever having to solve them explicitly.

In short, the methods we provide are practical and powerful and make short work of finding non-supersymmetric vacua and their properties in these flux systems. In slightly more technical language, we propose to re-formulate the necessary calculations arising from the extremisation of the potential (and, indeed, extremisation problems at large) in terms of **algorithmic algebraic geometry** and **commutative algebra**.

We will show the reader that the flux stabilisation problem generically translates to the study of saturation and primary decomposition of certain radical ideals in polynomial rings over appropriate ground algebraic fields. This rephrasing is far from a need for sophistry, but, rather, instantly allows effective algorithms, most of which have been implemented in excellent computer packages such as [10, 11], to be applied. In fact, we show that the quantities of physical interest are associated with the real roots of complex algebraic varieties. Once the affine variety of interest has been processed using the above complex methods, the information of physical relevance can be extracted using real algorithmic algebro-geometric techniques. In particular, we make extensive use of real root counting and sign condition routines based on the theory of Sturm queries. These algorithms then provide us with tremendous amounts of physical information about the vacua of $\mathcal{N} = 1$ moduli theories.

The methods we present find their most natural application within the context of perturbative stabilisation mechanisms, viz., potentials descending from form fluxes, torsion and non-geometric effects. In the interests of brevity and clarity we therefore concentrate on such cases in this paper. Practically, if one is interested in completely stabilised geometric vacua this would imply the consideration of models in type IIA or G_2 structure compactifications of M-theory. Non-perturbative effects *can* however be included in this type of analysis and we describe how this can be achieved later on in the paper.

Our approach is very much in the spirit of [12] where a programme of systematically and algorithmically determining the moduli space of $\mathcal{N} = 1$ gauge theories, and in particular to look for hidden geometric structure in the MSSM, was initiated. Here, we go one step further in our computational capability and utilise versatile and productive algorithms in both complex and real computational geometry and ideal theory.

The paper organised as follows. We begin in Section 2 by translating the computation of perturbative moduli stabilisation to one of algorithmic algebraic geometry. The problem of finding different vacua, SUSY, non-SUSY, Minkowski, AdS, etc., is classified by the type of physical questions with which one is faced. We show in pedagogical detail why one is led to the study of ideals, their radicals, as well as primary and saturation decompositions. Throughout we will focus on the precise algorithms needed for the investigations at hand and how they are used in conjunction with one another. At the end of section 2 we recover the physical classification presented at the start in a more mathematical context. It arises naturally in the process of organising the problem so that it is susceptible to the methods of algorithmic algebraic geometry.

In Section 3, a first example of the utility of the methods we espouse is provided. Using a model taken from the literature on non-geometric compactifications [13], we show how the concepts of resultants and their multi-variate generalisation, as well as elimination-order Gröbner bases, provide us with various constraints which flux parameters must satisfy in such models for there to be vacua with various properties.

In Section 4, we illustrate the various methods described in Section 2 for algorithmically finding flux vacua and their properties. This is achieved by applying our methods to a sample of problems drawn from the literature, ranging from compactifications of M-theory to type II and heterotic string theories. It is demonstrated that indeed the algorithmic methods described constitute a conducive path for research in the field, of diverse applicability. Finally, we conclude in Section 5. To make the paper self-contained we have included an extensive Appendix as a quick guide, first to algebraic geometry and theory of polynomial ideals, and second to the actual algorithms in complex and real geometry and commutative algebra used throughout the paper.

2 Flux Vacua and Algebraic Geometry

We wish to study four dimensional supergravity theories. In the context of moduli stabilisation, where the chiral superfields of interest are neutral under any gauge group, such theories are specified[1] by a Kähler potential K, and a superpotential W. The K and W which arise in such string and M-theory phenomenological contexts are not arbitrary. Both quantities generically take on certain general forms which are common to all of the perturbative stabilisation mechanisms currently being investigated in the literature. As such we shall concentrate on theories with this structure.

First, we require that the Kähler potential be taken as a sum of logarithms of (non-holomorphic) polynomials in the fields. This class of theories includes the standard form seen in the large volume and complex structure limits of string and M-theory compactifications of phenomenological interest. These limits are normally considered in discussions of moduli stabilisation so that the use of an effective supergravity is justified, and so that explicit polynomial formulas can be obtained respectively. We shall briefly describe how to extend our methods to other regions of complex structure space later. A typical form for the Kähler potential of such a system is as follows:

$$\begin{aligned} K &= -\log(S+\bar{S}) - \log(d_{ijk}(T^i+\bar{T}^i)(T^j+\bar{T}^j)(T^k+\bar{T}^k)) \\ &\quad -\log(\tilde{d}_{ijk}(Z^i+\bar{Z}^i)(Z^j+\bar{Z}^j)(Z^k+\bar{Z}^k)) \ . \end{aligned} \tag{3}$$

Here d and $\tilde{d}$ are constants, which could be related to the intersection numbers of the Calabi-Yau threefold and its mirror in the case of an $SU(3)$ structure compactification without intrinsic torsion for example. For the discussion at hand such constants will be regarded as mere constant parameters; their origin will not be important.

Next, we must specify the superpotential W. In the same limits of large complex structure, volume and weak coupling we again see a common form arising for the perturbative superpotentials which are found in moduli stabilisation contexts. The superpotential takes the form of a holomorphic polynomial in the fields. This kind of superpotential includes all of the perturbative stabilisation mechanisms known to date: flux, geometrical intrinsic torsion, and non-geometric elements in the compactification manifold. For example, the superpotential obtained for the heterotic string with fluxes on an generalised half-flat manifold is given as follows [14, 15]:

$$\begin{aligned} W &= -i(\epsilon_0 - iT^i p_{0i}) + (\epsilon_a - iT^i p_{ai})Z^a + \frac{i}{2}(\mu^a - iT^i q_i^a)\tilde{d}_{abc}Z^b Z^c \\ &\quad + \frac{1}{6}(\mu^0 - iT^i q_i^0)\tilde{d}_{abc}Z^a Z^b Z^c \ . \end{aligned} \tag{4}$$

[1] For the reader interested in charged fields we note that D-terms can be included trivially in the discussion that follows. For the sake of brevity we shall not, therefore, mention them further.

Here the ϵ's and μ's are parameters describing the fluxes present in the compactified space, while the p's and q's describe the intrinsic torsion.

Non-perturbative contributions to the superpotential of course will not take the form of a polynomial such as (4). The simplest implementation of the techniques we will shortly describe requires the superpotential to be polynomial in the fields. Given the possibility of complete perturbative stabilisation in some models we shall adhere to this case for the present. Later, we shall return to the issue of non-perturbative contributions to the superpotential where we shall describe how these may be accommodated within the structure we advocate.

Given the above Kähler and superpotentials one can proceed, for uncharged moduli fields, to construct the scalar potential from the usual formulas [16]. The scalar potential is given by:

$$V = e^K \left[\mathcal{K}^{A\bar{B}} D_A W D_{\bar{B}} \bar{W} - 3|W|^2 \right] \ . \tag{5}$$

As usual the D_A represents the Kähler derivative $\partial_A + \partial_A(K)$ and $\mathcal{K}^{A\bar{B}}$ is the inverse of the field space metric

$$\mathcal{K}_{A\bar{B}} = \partial_A \partial_{\bar{B}} K \ . \tag{6}$$

Given the above-mentioned forms of the Kähler potential and the superpotential, the potential is a quotient of polynomials in the fields. This feature, together with the polynomial nature of W, will be crucial to the methods which we will utilise throughout this paper. We note that the potential can still be written as such a quotient even when raising terms such as those added in [4] are included.

In the problem of moduli stabilisation, we are interested in finding the extrema, and in particular the minima, of the potential (5). In addition to the supersymmetric minima commonly discussed in the literature, for which $DW = 0$, this will in general include non-supersymmetric vacua. These vacua can be de Sitter or Minkowski even in the absence of D-terms or any other "raising" mechanisms. Non-supersymmetric minima of this type are not normally considered in the literature as even in simple models they are extremely difficult to find - a point to which we shall return shortly. The other extrema of the potential are also of some interest. The position of maxima neighbouring stabilised vacua, for example, might tell us about which set of cosmological initial conditions will allow the system to obtain the stabilised configuration. Likewise, such information can make it possible to estimate the rate of decay of a metastable vacuum due to tunneling.

2.1 Classification of the Problem

For clarity, it is expedient to classify the problem at hand into the following subtypes, each of which shall be addressed in turn in the ensuing sections. Let there be n fields indexed by i, then, the extremisation problem requires that

$$\partial_i V = 0, \quad \text{for } i = 1, \dots, n \; . \tag{7}$$

We can classify the solutions to (7) by the amount of supersymmetry they preserve, the value of the bare cosmological constant they dictate and so forth. We find it useful to define the following four subtypes:

SUSY, Minkowski	$D_i W = 0, \ \forall i, \quad W = 0$
SUSY, AdS	$D_i W = 0, \ \forall i, \quad W \neq 0$
NON-SUSY, Partially F-flat	$D_i W = 0, \ i = 1, \dots, m < n$
NON-SUSY, Non F-flat	$D_i W \neq 0 \ \forall i$

(8)

Now, recall that our potential is a rational function in the fields. As such, the first derivatives of the potential can also be written as quotients of polynomials with a related denominator. Physically, we are not interested in the solutions to the resulting equations which are given by taking the denominator to infinity. These correspond to the infinite field runaways common to these models. Therefore, it suffices to confine ourselves to the cases where the numerators of the first derivatives of the potential vanish.

In conclusion then, all the four subtypes of problems in (8) deal with the vanishing of systems of multi-variate (non-holomorphic) polynomial equations. To further simplify we circumvent the issue of the presence of both holomorphic and anti-holomorphic terms by substituting the expressions for the fields in terms of their real and imaginary parts. This then reduces the problem to that of finding the real roots of systems of complex polynomials. It should be noted that the problem can also be reduced to such a form in the presence of matter, where one would expand the potential up to some given order in these extra fields as usual.

2.2 Mapping the Problem to Algebraic Geometry

One can try and analytically solve the equations prescribed in (7) and (8). This can be quickly seen to be impossible in all but the most trivial cases. The reason for this is that, even if one of the polynomials is of a sufficiently low degree in a given variable to allow for an analytical solution, when one substitutes this solution back into the equations to obtain a system for the remaining variables the degree of this system with respect to the other degrees of freedom is increased. In a very small number of steps the remaining variables all appear with degree five or higher and the system can not be solved. Indeed, solving systems of multivariate polynomial equations is notoriously difficult.

Numerical techniques do not seem to fair any better. Locating the desired minima with such methods is intrinsically difficult due to the shallow nature of the minima and the strong features generically present elsewhere in the potential. Furthermore, minima of the types desired will in general only appear for certain parameter values and this would result in a very laborious system of trial and error attempts to find suitable values. We are compelled, therefore, to seek more effective methods.

Our extrema are defined by the vanishing of a set of complex polynomials in the (real) fields. Let us temporarily allow the real fields to take complex values. This results in the submanifold of (complexified) field space which corresponds to the extrema being defined as the locus where a collection of holomorphic polynomials vanish. This is the definition of a complex algebraic variety. The reader unfamiliar with algebraic geometry is directed to the Appendices where, to make the paper as self-contained as possible, the necessary concepts and constructions are provided. Our moduli stabilisation problem is then to *find the loci of real roots of a complex variety.* As described in appendix A any given affine variety can be described by ideals in a complex polynomial ring. The extremisation problem of (7) dictates that our variety must be defined by an ideal which is generated by the numerators of the first derivatives of the potential V. We shall denote this ideal by $\langle \partial V \rangle$.

As a technical point, multiple ideals describe the same variety. For example, as far as the physics is concerned, $\langle x \rangle$ and $\langle x^2 \rangle$ describe the same variety, even though the ideals themselves as sets of polynomials differ. To neglect such subtle scheme-theoretic differences, one can use the so-called radical ideal, which essentially removes trivial powers of the elements of the ideal. We denote the radical ideal obtained from $\langle \partial V \rangle$ as $\sqrt{\langle \partial V \rangle}$. To obtain the latter from the former, one can use a standard algorithm [17] as implemented in [10, 11].

Now that we have stated our problem in terms of algebro-geometrical language we may proceed to use some of the powerful techniques which have been developed in that field to advance our analysis. For clarity of notation let us first tabulate the key symbols which will be used throughout; these will be explained in detail in Appendix A.

2.2.1 Nomenclature

- $I := \langle f_1, \ldots, f_n \rangle$ denotes an ideal generated by polynomials $f_1, \ldots, f_n$.

- $L(I)$ denotes the variety corresponding to the ideal I and $I(M)$ denotes the ideal corresponding to the variety M. There is reverse-inclusion in the sense that $L(I \cup J) = L(I) \cap L(J)$ and $L(I \cap J) = L(I) \cup L(J)$.

- $\sqrt{I}$ denotes the **radical** of ideal I. Hilbert's Nullstellensatz is the statement on the geometry-algebra correspondence: $I(L(J)) = \sqrt{J}$.

- The **quotient** of ideal I by J is denoted $(I : J)$. Closely related is the **saturation** of I by J, denoted as $(I : J^\infty)$, corresponding geometrically to the sublocus of $L(I)$ which does not intersect $L(J)$.

2.3 Techniques from Complex Algebraic Geometry

We have now a mathematical object defining the space of extrema of the potential: it is the variety $L(\sqrt{\langle \partial V \rangle})$ corresponding to the ideal $\sqrt{\langle \partial V \rangle}$. This variety is not in general irreducible. Physically, this simply corresponds to the fact that the extrema of the potential may not be connected into one piece. There may be isolated minima and maxima, loci of minima with flat directions and so on. Mathematically, this means that $\sqrt{\langle \partial V \rangle}$ is not a prime ideal, but rather collectively contains information about all of the different extremal loci, the union of which is the extremal variety. Clearly it would be useful to be able to separate out the information about, say, lines of maxima, from that of isolated minima. Fortunately, a procedure exists in algorithmic algebraic geometry which does precisely this.

2.3.1 Primary Decomposition

It is a theorem that any radical ideal such as $\sqrt{\langle \partial V \rangle}$, as we are working over a polynomial ring over the complex numbers, is uniquely expressible as an irredundant finite intersection of prime ideals. Each prime ideal corresponds to an irreducible variety and physically represents a disconnected locus of extrema. The process of finding these prime ideals is a heavily studied subject in algorithmic algebraic geometry and is called **primary decomposition**. A number of algorithms have been developed to perform primary decomposition [18, 19, 20]. We shall make extensive use of the Gianni-Trager-Zacharias (GTZ) algorithm [18] later on in this paper when we come to analyse examples and as such a brief introduction to this is included in appendix B. This algorithm has been implemented in [11] by GTZ and Pfister.

If we denote the prime ideal describing the i-th locus by $P(i)$ then, we have the following.

$$\sqrt{\langle \partial V \rangle} = P(1) \cap P(2) \cap \ldots \cap P(k) \ . \tag{9}$$

Here k is the number of irreducible components of the extremal variety - the number of different loci. The prime ideals $P(i)$ are in general much simpler objects than the reducible $\sqrt{\langle \partial V \rangle}$. As such this process, even on its own, can be of considerable use in attacking problems of our kind. This will be seen explicitly once we move on to describe specific examples.

In summary, we can split up the extremisation problem of (7) by performing a primary decomposition of the radical $\sqrt{\langle \partial V \rangle}$ of the ideal $\langle \partial V \rangle$. The four subtypes of the problem according to (8) can, of course, be treated in the same way and we will shortly demonstrate this concretely.

2.3.2 Dimension and Flat-Directions

Once we have this series of prime ideals describing the various extremal loci for the potential of our flux system we can proceed to extract information about the various extrema. The extremal manifold $L(\sqrt{\langle \partial V \rangle})$, using the reverse-inclusion mentioned in Subsection 2.2.1 and (9), splits up into unions of irreducible pieces:

$$L(\sqrt{\langle \partial V \rangle}) = L(P(1)) \cup L(P(2)) \cup \ldots \cup L(P(k)) \ . \tag{10}$$

One of the most important things to know about a given locus of extrema is its dimension. Our chief interest will be in minima which are *isolated in field space*; these are fully stabilised vacua.

For an extremum i to be isolated, the dimension of the corresponding prime ideal $P(i)$, (or equivalently the dimension of $L(P(i))$) must be zero[2]. Physically, the piece $L(P(i))$ of the vacuum would then consists only of discrete points. In general, the i-th extremal locus $L(P(i))$ will not be zero-dimensional, and will exhibit flat-directions, the number of these are obviously dictated by the dimension of $P(i)$. We conclude that for all i,

$$\text{Number of Flat directions of locus } i = \dim(P(i)) \ . \tag{11}$$

Algorithms have been widely developed for computing the dimensions of ideals. A method for testing whether an ideal is zero dimensional, for example, is described in Appendix B.

Once we know the dimensions of the k prime ideals in the decomposition (9) we have then obtained significant physical information about our system. For example, if we were to find that none of the prime ideals are zero dimensional then that flux system would have no completely stabilised vacua without flat directions, either supersymmetric or non-supersymmetric. If some of the prime ideals are indeed zero dimensional, and if we are only interested in isolated vacua, we can then confine our attention to this subset of the full expansion (9).

[2]The alert reader may be concerned that we are talking about the dimension of a complex variety when physically we are interested in the dimension of the space of real roots. For a real root to be isolated it is a prerequisite that the complex dimension of the associated $P(i)$ is zero. If this is not the case we may simply vary the real part of one of the unconstrained complex fields. It may be the case however that a zero dimensional complex variety has no real roots. This is a question to which we will shortly return.

Now, we wish to go on to answer more detailed questions. In particular, we are interested in the following inquiries. If an ideal is zero dimensional do any of the corresponding extrema correspond to real field values? Are the resulting isolated extrema maxima, minima or saddle points? Are the extrema in a well controlled part of field space where we can trust the various approximations made in obtaining the low energy effective theory we have been studying? Do the extrema correspond to de sitter, anti de Sitter or Minkowski four dimensional universes? Are the extrema supersymmetric? To answer these questions we need to turn to the subject of real, as opposed to complex, algorithmic algebraic geometry. This is the subject of the next subsection.

2.4 Techniques from Real Algebraic Geometry

We now have some zero dimensional ideals $P(i)$ at hand. As discussed above, we ultimately wish to study the real roots of our polynomial system. We now show that it is possible to extract the physically relevant information about the extrema of the potential without ever finding the explicit location of these roots, in which we mostly have no interest in any event. This situation could be compared to the use of algebraic geometry in describing smooth Calabi-Yau compactifications. There, we do not know any explicit metric on the internal space yet we can still extract much of the physically relevant information.

As a brief remark, if we primary decompose over the complex numbers it is always possible to trivially solve any resulting zero dimensional prime ideals explicitly for the relevant roots. The algebro-algorithmic methods described below are still vital, however, for two reasons. First, actual implementations of primary decomposition algorithms normally work over the rationals where it is not so clear that finding explicit solutions of zero dimensional primes is always possible (although we have found in practice it is for these systems - an unexpected bonus!). Second, these algorithms can reduce the number of costly primary decomposition calculations we have to perform in analysing a system. These comments will be illustrated concretely in later sections.

Indeed, each polynomial system $P(i)$, can be, by expanding all of the coefficients into their real and imaginary parts (or by working over the rationals from the start - which is what we do in practice), turned into a system in $\mathbb{R}[x_1, \ldots, x_n]$. We are thus entering the realm of real algebraic geometry. In particular, we need to know about the real roots of real polynomial ideals. Much less is known about this field than about its complex cousin. However, it turns out that some of the few algorithms currently available furnish us with exactly the tools we require to extract what we wish to know.

2.4.1 Sign Conditions and Real Root Finding

We will make extensive use of two kinds of algorithms [21]. The first kind allows us to compute the number of real roots of a zero dimensional ideal (i.e., it allows us to find the number of physical isolated extrema of our potential). The second allows us to compute the signs of any given set of polynomial functions on each of the real roots of the system [21], by means of a so-called Sturm query. A brief description of how these algorithms work is provided in Appendix C. Both of these kinds of algorithm have been implemented in [11] by Tobis [22].

We proceed then by using the first of these algorithms to find the number of real isolated extrema of our potential. We then go on to use the second to extract the relevant physical information about these extrema.

Stability of the vacua: The double derivatives of the potential with respect to the fields for the system specified in equations (3) and (4) take the form of quotients of polynomials which make up the Hessian matrix $\frac{\partial^2 V(x)}{\partial x_i \partial x_j}$. In order to check the character of the extremum one can compute the characteristic polynomial of this Hessian matrix (which is, in fact, a rational function) and focus on its numerator polynomial. We can then form the ideal generated by the characteristic polynomial and the zero-dimensional primary ideal, describing a solution branch and perform an appropriate series of Sturm queries on its roots. This allows you to decide algorithmically whether the extremum is a minimum, maximum or saddle point.

Due to the effect pointed out by Breitenlohner and Freedman [23] it is necessary to determine whether these extrema are de Sitter, anti de Sitter or Minkowski before we can say whether they correspond to stable vacua. If an extremum is a minimum or saddle point with negative cosmological constant it could still be stable. To discover whether this occurs in any given case one must check the sign of a certain set of functions [23]. In fact, as phrased in [15], the bound one needs to test, at the critical point x_0 of the potential V, is determined by the matrix

$$\left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j} - \frac{3}{2} V(x) \mathcal{K}_{ij}(x) \right) \Bigg|_{x_0} . \tag{12}$$

If the eigenvalues of this matrix are all non-negative, then the AdS minimum is stable. Indeed, for Minkowski or dS, the positive-definiteness of the Hessian matrix $\frac{\partial^2 V(x)}{\partial x_i \partial x_j}$ suffices for stability of the minimum. In our case these tests again all turn out to be quotients of polynomials and so this can be achieved with the aforementioned algorithms. We can therefore determine how many completely stabilised vacua the system has.

Validity of the effective theory: For these vacua to be in a regime in which our supergravity description is valid we need the values of certain fields, the size of the internal space for example, to be much bigger than 1 - let us say greater than 10. By checking the sign of the polynomial $t - 10$, where t is the field under consideration, we can check whether this is the case for each of our stabilised vacua.

Geometry of the vacua: The potential of the system is, as we have already pointed out, a quotient of polynomials. As such to deduce whether our extrema correspond to Minkowski, anti de Sitter or de Sitter spacetimes it suffices to again find the sign of the numerator and denominator.

Supersymmetry of the vacua: Another important piece of information to have is whether the vacua are supersymmetric or not. The F-terms of our system, given (3) and (4), are again rational functions and so we can check their sign on each of our stabilised, controlled extrema. In particular, the algorithms described in Appendix C will tell us if these polynomials vanish. We can thus determine which of the stabilised vacua are supersymmetric and which are not.

In conclusion, we can learn essentially all of the important information we require about the vacua, both supersymmetric and non-supersymmetric, completely algorithmically, without ever having to explicitly solve the system. Many of the interesting properties of the particle physics associated with each vacuum can also be ascertained in this manner. The perturbative contributions to the masses and Yukawa couplings in these models, for example, are rational functions of the moduli (in appropriate limits). These points raised above clearly constitute a very interesting set of questions. We will now pause our general discussions and proceed to show how such questions may be attacked concretely.

2.5 Saturations and Classification Revisited

Having tantalised the readers, we now point out a caveat emptur lest they are overwhelmed with optimism. In practice, the above discussion has limits when pursued using the prepackaged implementations of the algorithms available in such computer programs as [10, 11]. Indeed, naive applications of the programs often cause them to struggle, halt, or run out of memory.

There are, luckily, various known tricks for avoiding this set of affairs [24]. These tricks all fall under the philosophy of **splitting principles** and are concerned with splitting the problem up into more manageable pieces, even before passing the problem to a primary decomposition algorithm.

One key notion in these so-called splitting principles is the idea of a **saturation decomposition**. In this subsection, we will see how this seemingly esoteric technique precisely adapts itself to our goal. A more detailed definition and discussion of saturations can be found in appendix A. Briefly, given an ideal I and a polynomial f, the saturation, denoted $(I : f^\infty)$, is equal to

$$sat(I, f) := (I : f^\infty) = \bigcup_{n=1}^{\infty} (I : f^n) \ , \tag{13}$$

where each $(I : f^n)$ is the quotient of I by f^n, which is discussed in detail in Appendix A. The point is that the saturation $(I : f^\infty)$ corresponds geometrically to the space of all zeros of the ideal I for which the polynomial f does not vanish [3].

We now follow the idea in [24] to utilise the splitting principle. Suppose, for some integer l, the following identity holds:

$$(I : f^\infty) = (I : f^l) \ . \tag{14}$$

In other words, at some finite l the quotient has removed all powers of f from I. Then, we have the following decomposition of the ideal I:

$$I = (I : f^\infty) \cap \langle I, f^l \rangle \ , \tag{15}$$

where $\langle I, f^l \rangle$ is the ideal generated by I together with f^l. If we take the radical to neglect powers, then we have

$$\sqrt{I} = \sqrt{(I : f^\infty)} \cap \sqrt{\langle I, f \rangle} \ . \tag{16}$$

[3]In fact to be precise the saturation defines geometrically the *closure* of the complement of $L(f)$ in $L(I)$. If I is one dimensional then there may be zero dimensional points in the variety associated to the saturation for which $f = 0$ for example. In the bulk of this paper, when we will be interested in using saturations, our primary concern will be with zero dimensional ideals where this subtlety does not arise.

Geometrically, (16) is the split we desire: it says that $L(I)$ is the union of a subvariety $L(\sqrt{(I : f^\infty)})$ where f does not vanish, with a subvariety $L(\sqrt{\langle I, f\rangle})$ where f does vanish.

We pause to ask, what is a good choice of polynomial f, or, iteratively, a set of such f's? In general, finding a non-trivial zero divisor, an element f for which $(I : f) \neq I$, can be very difficult. For the problem at hand, however, our supersymmetric theories automatically provide the perfect choice! *These f's are simply the F-flatness conditions.* Recall that one of our problems from (8), the partial F-flat case (which computationally is the most illustrative case), is to find the solutions to $\langle \partial V\rangle$ such that $f_i = D_i W$ (or, strictly, the polynomial numerators of $D_i W$) vanishes only for a subset of fields $i = 1, \ldots, m < n$. We therefore, naturally, choose each F-flatness equation as an f, iterating from $m + 1$ to n. Geometrically, we can write this saturation decomposition of the vacuum manifold as:

$$
\begin{aligned}
L(\partial V) \;=\; & L(\langle \partial V, f_1, f_2, ..., f_n\rangle) \cup \\
& \bigcup_i L((\langle \partial V, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n\rangle : f_i^\infty)) \cup \\
& \bigcup_{i,j} L\Big(\big((\langle \partial V, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n\rangle : f_i^\infty) : f_j^\infty\big)\Big) \cup \\
& \vdots \\
& L\Big(\big(\big(\ldots(\partial V : f_1^\infty)\ldots : f_{n-1}^\infty\big) : f_n^\infty\big)\Big) \ .
\end{aligned} \tag{17}
$$

In words, what this decomposition describes is a classification of the different possible vacua according to how many of the F-flatness conditions they obey. Thus the first term here is simply the supersymmetric vacuum space. The second term is the union of all the vacuum spaces for which only one of the F-flatness equations is disobeyed, and so on. Once one has broken up the problem in this manner one can go on to apply the analysis discussed in previous subsections.

Therefore, this decomposition is physically intuitive, and natural from the point of view of the theory of ideals, as well as being practically useful. The classification (8) corresponds precisely to (17). The Minkowski vacuum, for example, would be a subset of the first term, given by $L(\langle \partial V, f_1, \ldots, f_n, W\rangle)$, where the superpotential W vanishes in addition to all of the F-flatness conditions. Here, a further simplification can be made; indeed, F-flat configurations are automatically extrema of the potential in supersymmetric systems. Thus, the Minkowski vacuum is then $L(\langle f_1, \ldots, f_n, W\rangle)$,

If we wish to study a given type of vacuum - be it partially F-flat, non-F-flat or completely F-flat, all we have to do is to perform the associated saturation decomposition in (17). Working with each of these pieces is much more tractable than working with $\langle \partial V\rangle$ in its entirety. Indeed, some information can be extracted immediately after forming these saturations. For example, if a given piece in the saturation decomposition has a dimension of -1 (this is the convention that the system has no roots) then the associated set of vacua are absent in the model under consideration.

We have come full circle and, in the course of setting up a practical method for finding minima, have recovered the physical classification (8) in the more mathematical context of (17). We shall stop our general discussion here. In the following sections, we will address each of the subtypes discussed in (8), by illustrating with actual examples taken from string and M-theory phenomenology. In these specific examples we will find that our method is indeed powerful. Primary decomposition breaks the original extremely complicated sets of polynomial equations up into more manageable pieces. The prime ideals containing the completely stabilised vacua are so much simpler than the full system that they can be often solved explicitly - thus furnishing us with a complete knowledge of the vacua we find.

Let us then proceed to analyse various parts of this expansion for a variety of models. Our aim in doing this will be to illustrate the power of this methodology, as well as to see what general statements can be extracted in each case.

3 The SUSY Minkowski Case and Constraints on Flux

Let us begin with the case of supersymmetric Minkowski vacua. Here, we are solving for the vanishing of the superpotential and its derivatives. In this case, some general theory can be developed and general, necessary and sufficient, conditions on the fluxes for the existence of such vacua can be derived. Similar constraints can be derived in the other cases but in those instances these are only necessary conditions. Necessary conditions for the existence of non-supersymmetric Minkowski minima in supergravity have also been given in [25].

3.1 Resultants and Diophantine Equations

Before embarking on a full discussion, let us see what happens if there were only a single (complex) field. That is, W is a degree n polynomial of a single variable x with integer coefficients determined by the values of the fluxes. We are therefore solving the system

$$\begin{aligned} W(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0 \\ W'(x) &= a_1 + 2a_2 x + \ldots + n a_n x^{n-1} = 0\ . \end{aligned} \tag{18}$$

Already, one can learn quite a lot. We know that two univariate polynomials have common zeros iff their resultant vanishes [26]. Therefore, we require that

$$
\begin{aligned}
&\mathrm{res}(W(x), W'(x)) = \\
&\det \begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_1 & a_0 & 0 & \dots & 0 \\
0 & a_n & a_{n-1} & a_{n-2} & \dots & a_2 & a_1 & a_0 & 0 & \dots \\
\vdots & & & \vdots & & & & & & \vdots \\
\vdots & & & (n-1) \text{ times} & & & & & & \vdots \\
\vdots & & & \vdots & & & & & & \vdots \\
na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & (n-3)a_{n-3} & \dots & 2a_2 & a_1 & 0 & \dots & 0 \\
0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 3a_3 & 2a_2 & a_1 & 0 & \dots \\
\vdots & & & \vdots & & & & & & \vdots \\
\vdots & & & (n) \text{ times} & & & & & & \vdots
\end{pmatrix} \\
&= 0 \ .
\end{aligned}
\tag{19}
$$

In general, the resultant of an order m polynomial with an order n one is homogeneous of degree $m+n$ in the coefficients. In other words, for our case, the determinant in (19) is a polynomial in the a_i, of homogeneous degree $2n-1$. This is easy to see. Each element in the matrix in (19) is either 0 or one of the coefficients. Each term in the determinant, when expanded, receives one factor from each column. Any non-vanishing term then has the same degree as the diagonal term, which is $a_n^{n-1}a_1^n$, of degree $2n-1$. This seemingly trivial observation has interesting consequences. It dictates that the resultant vanishes, if and only if the coefficients satisfy a homogeneous Diophantine equation.

Now, recall that in the general problem of studying the critical points of the (ordinary) potential there are holomorphic and anti-holomorphic fields in our defining polynomials and we needed to expand them into their real and imaginary components and look for real roots corresponding thereto. However, our Minkowski problem is simpler in that we need only studying the vanishing of the (holomorphic) superpotential and its derivatives, and it suffices to find complex roots of a purely holomorphic polynomial system as above. Therefore, we can conclude that *a Minkowski vacuum exists iff the resultant, a homogeneous Diophantine equation in the fluxes, vanishes.* Of course, there is nothing to guarantee that such vacua would be physical in the sense that the values of the real parts of the superfields would be large and so forth. To check whether this is the case one would have to utilise the methods detailed in the ensuing section.

As an illustration, let us present the resultant explicitly for some small values of n:

n	resultant	degree
1	a_1	1
2	$a_2\,(-a_1^2 + 4\,a_0\,a_2)$	3
3	$a_3\,(-{a_1}^2\,{a_2}^2 + 4\,{a_1}^3\,a_3 - 18\,a_0\,a_1\,a_2\,a_3 + a_0\,(4\,{a_2}^3 + 27\,a_0\,{a_3}^2))$	5
4	$a_4\,(-27\,{a_1}^4\,{a_4}^2 + {a_1}^3\,(-4\,{a_3}^3 + 18\,a_2\,a_3\,a_4) - 2\,a_0\,a_1\,a_3\,(-9\,a_2\,{a_3}^2$ $+40\,{a_2}^2\,a_4 + 96\,a_0\,{a_4}^2) + {a_1}^2\,({a_2}^2\,{a_3}^2 - 4\,{a_2}^3\,a_4 - 6\,a_0\,{a_3}^2\,a_4$ $+144\,a_0\,a_2\,{a_4}^2) + a_0\,(-4\,{a_2}^3\,{a_3}^2 + 16\,{a_2}^4\,a_4 + 144\,a_0\,a_2\,{a_3}^2\,a_4$ $-128\,a_0\,{a_2}^2\,{a_4}^2 + a_0\,(-27\,{a_3}^4 + 256\,a_0\,{a_4}^3)))$	7

(20)

It would be interesting to study the solutions to such Diophantine equations. The foundational work on this subject is laid out in [27], with some recent surveys and results in [28, 29].

3.2 Multi-variate Resultants and an Example

We have discussed the univariate situation above. What about the general case where there is more than one variable? In multivariate examples the equivalents of the resultant of the previous subsection can be computed algorithmically using an elimination order Gröbner basis [30, 31]. In other words, there is a systematic method of eliminating variables stepwise from an ideal, just like Gaussian elimination for linear systems. A description of the algorithm for calculating a Gröbner basis in the lexicographic ordering - which is an example of an elimination ordering - is provided in Appendix A. This elimination, in the uni-variate case, produces the resultant discussed in the previous subsection.

Algebraically this process takes the intersection $I \cap \mathbb{C}[X_1, ..., X_n]$, of the original ideal $I \subset \mathbb{C}[X_1, ..., X_n, a_1, ..., a_m]$ (where the X's are the variables and the a's the parameters in the original problem) with the ring $\mathbb{C}[X_1, ..., X_n]$ of variables to be eliminated. Geometrically, this simply corresponds to the projection of the original ideal on to the subspace of the space described by the original ring where the eliminated variables vanish. The resultant conditions on the a's are then clearly necessary and sufficient for the existence of a root of I for some value of the X's.

Thus, even in the multivariate case, constraints on the fluxes which are necessary and sufficient conditions for the existence of supersymmetric Minkowski vacua can still be found. In some cases these can be quite compact in form. In others, however, the resulting constraint equations can be quite appreciable in size, as we shall see in a concrete example now. This constraint on the practicality of resultants of multivariate systems above a certain level of complexity is well known [27].

Let us illustrate with a concrete example from the literature. Take equation (2.6) of [13], which presents a non-geometric flux superpotential of the form

$$
\begin{aligned}
W &= a_0 - 3a_1\tau + 3a_2\tau^2 - a_3\tau^3 \\
&\quad +S(-b_0 + 3b_1\tau - 3b_2\tau^2 + b_3\tau^3) \\
&\quad +3U(c_0 + (\hat{c}_1 + \check{c}_1 + \tilde{c}_1)\tau - (\hat{c}_2 + \check{c}_2 + \tilde{c}_2)\tau^2 - c_3\tau^3),
\end{aligned} \tag{21}
$$

with the following constraints on the fluxes.

$$
\begin{gathered}
a_0b_3 - 3a_1b_2 + 3a_2b_1 - a_3b_0 = 16 \\
a_0c_3 + a_1(\check{c}_2 + \hat{c}_2 - \tilde{c}_2) - a_2(\check{c}_1 + \hat{c}_1 - \tilde{c}_1) - a_3c_0 = 0 \\
\begin{aligned}
c_0b_2 - \tilde{c}_1b_1 + \hat{c}_1b_1 - \check{c}_2b_0 &= 0 & \quad c_0\tilde{c}_2 - \check{c}_1^2 + \tilde{c}_1\hat{c}_1 - \hat{c}_2c_0 &= 0 \\
\check{c}_1b_3 - \hat{c}_2b_2 + \tilde{c}_2b_2 - c_3b_1 &= 0 & \quad c_3\tilde{c}_1 - \check{c}_2^2 + \tilde{c}_2\hat{c}_2 - \hat{c}_1c_3 &= 0 \\
c_0b_3 - \tilde{c}_1b_2 + \hat{c}_1b_2 - \check{c}_2b_1 &= 0 & \quad c_3c_0 - \check{c}_2\hat{c}_1 + \tilde{c}_2\check{c}_1 - \hat{c}_1\tilde{c}_2 &= 0 \\
\check{c}_1b_2 - \hat{c}_2b_1 + \tilde{c}_2b_1 - c_3b_0 &= 0 & \quad \hat{c}_2\tilde{c}_1 - \tilde{c}_1\check{c}_2 + \check{c}_1\hat{c}_2 - c_0c_3 &= 0\,.
\end{aligned}
\end{gathered} \tag{22}
$$

There are also additional constraints which take the same form as those above but with the hats and checks switched around. Various useful pieces of algebraic processing of these constraints are provided in [13]. These relations come from, for example, tadpole cancellation conditions and integrability conditions on Bianchi identities.

Finding Minkowski vacua of this system is then the problem of studying the ideal $I = \{W, \partial_\tau W, \partial_S W, \partial_U W\}$ in the ring $\mathbb{C}(a_{0,1,2,3}, b_{0,1,2,3}, c_{0,1,2,3})[S,T,U]$, which is a polynomial ring in variables S, T and U but with all fluxes treated as parameters (formally, we call $\mathbb{C}(a_{0,1,2,3}, b_{0,1,2,3}, c_{0,1,2,3})$ an algebraic extension of the ground field $\mathbb{C}$). If one uses an implementation of the relevant algorithms in a package such as [10, 11] then it is assumed that none of the flux parameters vanish. The Gröbner basis of I in lexicographic order then immediately gives that I has negative dimension. In other words, there are no roots in I. This is a quite powerful statement without ever solving for anything, or even imposing the constraints (22): there are no Minkowski vacua for this model, if all of the parameters are non-vanishing.

Of course, some flux parameters can vanish. So let us treat them as variables and place I in an elimination order Gröbner basis, and eliminate S, T, U to obtain our constraints as described above.

The full result for the superpotential given in (21) can be obtained in a matter of seconds[4]. The result is a system of 28 constraint equations which the fluxes must obey. We do not

[4] The best way to achieve this is to homogenise the problem, use a Hilbert driven global elimination order Gröbner basis calculation, and then dehomogenise again at the end. See [31] for details.

present these expressions explicitly here as they amount to 8 pages of expressions in this font size comprising of 6 degree 3, 12 degree 4, 8 degree 5, and 2 degree 6 polynomials.

To provide a concrete result in a presentable fashion, let us simplify by setting, for example, all $a_{0,3}$, $b_{0,3}$ and $c_{0,3}$ to 1. Indeed, there are still many solutions of (22) with this choice. Now, treat I as an ideal in $\mathbb{C}[S,T,U,a_1,a_2,b_1,b_2,\hat{c}_1,\check{c}_1,\tilde{c}_1,\hat{c}_2,\check{c}_2,\tilde{c}_2]$. We again proceed to eliminate S,T,U using an implementation of elimination orderings in [10, 11]. We find the following constraints as necessary and sufficient for the existence of Minkowski vacua[5]:

$$\begin{aligned}
0 &= 3a_2b_1 - 3a_1b_2 + a_2c_1 - b_2c_1 - a_1c_2 + b_1c_2, \\
0 &= 27b_1b_2^2c_1 + 9b_2^2c_1^2 - 27b_1^2b_2c_2 + 3b_2c_1^2c_2 - 9b_1^2c_2^2 - 3b_1c_1c_2^2 \\
&\quad -27b_1^3 + 27b_2^3 - 27b_1^2c_1 - 9b_1c_1^2 - c_1^3 + 27b_2^2c_2 + 9b_2c_2^2 + c_2^3, \\
0 &= 27a_1b_2^2c_1 + 9b_2^2c_1^2 - 27a_1b_1b_2c_2 + 9a_1b_2c_1c_2 - 9b_1b_2c_1c_2 + 3b_2c_1^2c_2 \\
&\quad -9a_1b_1c_2^2 - 3b_1c_1c_2^2 - 27a_1b_1^2 + 27a_2b_2^2 - 18a_1b_1c_1 - 9b_1^2c_1 - 3a_1c_1^2 - 6b_1c_1^2 - c_1^3 \\
&\quad +18a_2b_2c_2 + 9b_2^2c_2 + 3a_2c_2^2 + 6b_2c_2^2 + c_2^3, \\
0 &= 27a_1a_2b_2c_1 + 9a_2b_2c_1^2 - 27a_1^2b_2c_2 + 9a_1a_2c_1c_2 - 9a_1b_2c_1c_2 + 3a_2c_1^2c_2 \\
&\quad -9a_1^2c_2^2 - 3a_1c_1c_2^2 - 27a_1^2b_1 + 27a_2^2b_2 - 9a_1^2c_1 - 18a_1b_1c_1 - 6a_1c_1^2 - 3b_1c_1^2 - c_1^3 \\
&\quad +9a_2^2c_2 + 18a_2b_2c_2 + 6a_2c_2^2 + 3b_2c_2^2 + c_2^3, \\
0 &= 27a_1a_2^2c_1 + 9a_2^2c_1^2 - 27a_1^2a_2c_2 + 3a_2c_1^2c_2 - 9a_1^2c_2^2 - 3a_1c_1c_2^2 - 27a_1^3 + 27a_2^3 \\
&\quad -27a_1^2c_1 - 9a_1c_1^2 - c_1^3 + 27a_2^2c_2 + 9a_2c_2^2 + c_2^3 \ .
\end{aligned} \tag{23}$$

Here, $c_1 = \hat{c}_1 + \check{c}_1 + \tilde{c}_1$ and $c_2 = \hat{c}_2 + \check{c}_2 + \tilde{c}_2$. To see, therefore, whether there are any Minkowski vacua for the choice of flux values mentioned above, we need only check whether the ideal formed by joining (23) and (22) over ground field $\mathbb{Z}$ has dimension zero or not.

In fact, in the system specified in (21) and (22) the Minkowski vacua always exhibit at least one flat direction even when present. It is easy to show that the curve given below defines a flat direction, in the (S,U) plane, of the potential obtained from (21) for any Minkowski vacuum.

$$-3a_1 + 6a_2\tau_0 - 3a_3\tau_0^2 + S(3b_1 - 6b_2\tau_0 + 3b_3\tau_0^2) + 3U(c_1 - 2c_2\tau_0 - 3c_3\tau_0^2) = 0 \tag{24}$$

Here τ_0 is the expectation value of the other modulus in the vacuum.

We would like to emphasise that constraints on the fluxes such as those given above can be obtained in this manner for *any* of the cases specified in (8). To do this, one simply takes the relevant piece in the saturation decomposition (treating parameters as variables) and eliminates the fields as above. In other words, elimination orderings can provide us with

[5]As before one would have to check whether such vacua correspond to physically acceptable field values using techniques presented in the next section.

necessary conditions on the fluxes for any type of vacuum to exist. However, in other cases, due to our complexification of the real field space in order to make the relevant polynomials holomorphic, the resulting constraints are only necessary and not sufficient. This is simply because the implied roots of the ideal, if the constraints are satisfied, could correspond to complex values for the real and imaginary parts of our complex scalar fields. Such roots do not of course correspond to physical vacua. In addition, while supersymmetric Minkowski extrema are always minima (with the possibility of flat directions), other forms of extrema can be unstable and therefore not correspond to vacua.

Having discussed how constraints on fluxes can be derived using elimination orderings we shall now resume our main discussion. In the next subsection we revert to the question of finding vacua in flux systems according to the methods of Section 2.

4 Attacking the General Problem and Finding Vacua

With the above prelude on constraints and the Minkowski case finished, let us systematically address the question of finding vacua, including the partially F-flat and Non-F-flat cases, using our decomposition methods as discussed in Subsection 2.5. We shall consider the full expansion (17) in several examples in this section.

4.1 An Illustrative Example

Let us completely analyse a simple first example to illustrate our method. Suppose we had a four dimensional $\mathcal{N} = 1$ supergravity theory defined by the following Kähler and superpotential:

$$\begin{aligned} K &= -3\log(T_1 + \bar{T}_1) - 3\log(T_2 + \bar{T}_2), \\ W &= -T_1^2 - T_1 T_2 - T_2^2 + 10T_1 + 10T_2 - 100 \ . \end{aligned} \tag{25}$$

Even this simple example results in complicated equations. Defining $T_1 = t_1 + i\tau_1$ and $T_2 = t_2 + i\tau_2$, the extrema of the potential V is defined by the following:

$$\begin{aligned} 0 &= 25(t_1^4 + t_1^2(500 - 280t_2 + 37t_2^2 - 10\tau_1^2 - 10\tau_1\tau_2 - 7\tau_2^2) + 4t_1(-140t_2^2 + 7t_2^3 \\ &\quad +30(100 + \tau_1^2 + 2\tau_1\tau_2) + 3t_2(200 + \tau_1^2 + 4\tau_1\tau_2 + \tau_2^2)) - 3(20t_2^3 + t_2^4 - 60t_2(100 \\ &\quad +2\tau_1\tau_2 + \tau_2^2) + t_2^2(7\tau_1^2 + 10\tau_1\tau_2 + 10(-50 + \tau_2^2)) + 9(10000 + \tau_1^4 + 2\tau_1^3\tau_2 \\ &\quad -100\tau_2^2 + 2\tau_1\tau_2^3 + \tau_2^4 + \tau_1^2(-100 + 3\tau_2^2)))) \ , \\ 0 &= 25(18\tau_1^3 + 27\tau_1^2\tau_2 + \tau_2(5t_1^2 - 60t_2 + 5t_2^2 - 12t_1(5 + t_2) + 9\tau_2^2) + \tau_1(-900 + 10t_1^2 \end{aligned} \tag{26}$$

$$
\begin{aligned}
& +7t_2^2 - 6t_1(10+t_2) + 27\tau_2^2)) \ , \\
0 &= -25(3t_1^4 + t_1^3(60-28t_2) - t_2^4 - 120t_2(100+2\tau_1\tau_2+\tau_2^2) + t_1^2(-1500+560t_2 \\
& -37t_2^2 + 30\tau_1^2 + 30\tau_1\tau_2 + 21\ \tau_2^2) + t_2^2(7\tau_1^2 + 10\tau_1\tau_2 + 10(-50+\tau_2^2)) + 4t_1(70t_2^2 \\
& -45(100+\tau_1^2+2\tau_1\tau_2) - 3t_2(200+\tau_1^2+4\tau_1\tau_2+\tau_2^2)) \\
& +27(10000+\tau_1^4+2\tau_1^3\tau_2 - 100\tau_2^2 + 2\tau_1\tau_2^3 + \tau_2^4 + \tau_1^2(-100+3\tau_2^2))) \ , \\
0 &= 25(-60t_2(\tau_1+\tau_2) + 5t_2^2(\tau_1+2\tau_2) + t_1^2(5\tau_1+7\tau_2) - 6t_1(2(5+t_2)\tau_1 + t_2\tau_2) \\
& +9(\tau_1^3 + 3\tau_1^2\tau_2 + 3\tau_1\tau_2^2 + 2\tau_2(-50+\tau_2^2))) \ .
\end{aligned}
$$

Solving this system by conventional means is clearly impossible. According to our discussions, let us, instead, think of (26) as an ideal $\langle \partial V \rangle \in \mathbb{R}[t_1, t_2, \tau_1, \tau_2]$. We perform the saturation decomposition of (17) and present the components thereof in Table 1. We have 2 complex F-flatness equations: $F_{T_i} = D_{T_i} W = 0$, $i = 1, 2$. In the table and the text below we expand these into 4 real equations and take $\mathrm{Re}[F_{T_1}] = f_1$, $\mathrm{Re}[F_{T_2}] = f_2$, $\mathrm{Im}[F_{T_1}] = f_3$ and $\mathrm{Im}[F_{T_2}] = f_4$.

With the table we can begin our analysis. First we break up the ideals listed to extract any zero dimensional pieces. This part of the analysis [6] is performed using the factorising Gröbner basis routine [31] as implemented in [11]. Once we have a zero dimensional ideal we do not decompose it any further at this stage. Anything which is not zero dimensional, however, is primary decomposed to check whether it contains any zero dimensional factors. We are thus faced with a list of zero dimensional ideals; on these we check for two conditions that they must satisfy if they are to describe physical extrema:

1. The zero dimensional ideal should have real roots;

2. The real parts of our original superfields should be greater than 1 when evaluated at the extrema.

These checks are performed using the root counting and sign query algorithms based upon Sturm queries as implemented in [22, 11] and outlined in Appendix C.

The first condition is required because our ring variables correspond physically to the real and imaginary parts of the physical fields. The second condition is physically motivated. This kind of constraint is enforced in systems descending from flux compactifications so that the vacua concerned lie both in the large Kähler and large complex structure limits. Large values for the real parts of the equivalent of Kähler moduli in these situations are required

[6] This ancillary part of the process is not required in the algorithmisation of the problem of finding flux vacua and so was not mentioned in section 2. This is simply a practical point - some initial splitting up of the relevant ideals in this manner can make the, already quick, calculations involved much faster.

if the effective supergravity descriptions being used in these contexts is to be valid. Large complex structure generically leads to the relevant equations being polynomial in the fields. Indeed, more general cases can be dealt with in a similar way to non-perturbative effects (whose inclusion will be discussed later), at least in certain limits such as the conifold one.

Ideal	Interpretation as Vacua	Physical ?
$\langle f_1, f_2, f_3, f_4\rangle$	supersymmetric	Yes
$(\langle \partial V, f_i, f_j, f_k\rangle : f_l^\infty)$ where $i \neq j \neq k \neq l$ and $i, j, k, l = 1, \ldots, 4$	partially F-flat	No, $t_1 = 0$ or $t_2 = 0$
$((\langle \partial V, f_1, f_2\rangle : f_3^\infty) : f_4^\infty)$	partially F-flat	No, $t_2 = 0$
$((\langle \partial V, f_1, f_3\rangle : f_2^\infty) : f_4^\infty)$	partially F-flat	No, No real roots
$((\langle \partial V, f_3, f_2\rangle : f_1^\infty) : f_4^\infty)$	partially F-flat	No, No real roots
$((\langle \partial V, f_1, f_4\rangle : f_3^\infty) : f_2^\infty)$	partially F-flat	No, No real roots
$((\langle \partial V, f_4, f_2\rangle : f_3^\infty) : f_1^\infty)$	partially F-flat	Yes
$((\langle \partial V, f_3, f_4\rangle : f_1^\infty) : f_2^\infty)$	partially F-flat	No, $t_1 = 0$
$(((\langle \partial V, f_i\rangle : f_j^\infty) : f_k^\infty) : f_l^\infty)$ where $i \neq j \neq k \neq l$ and $i, j, k, l = 1, \ldots, 4$	partially F-flat	No, No real roots
$((((\partial V : f_1^\infty) : f_2^\infty) : f_3^\infty) : f_4^\infty)$	non-SUSY	No, No real roots

Table 1: Full saturation decomposition of the vacuum $\langle \partial V\rangle$ for the potential V and F-flatness equations f_i given in the example (26).

Examining Table 1 we see that we can confine our attentions to just two terms in the saturation expansion; the two physical ones, corresponding to the supersymmetric $\langle f_1, f_2, f_3, f_4\rangle$ and the partially F-flat $((\langle \partial V, f_4, f_2\rangle : f_3^\infty) : f_1^\infty)$ extrema; both are AdS. Indeed, as well as simplifying the analysis this allows us to make quite general statements. For example, all non-supersymmetric vacua of this system are partially F-flat with F_{T_2} always being zero in the vacuum. We proceed to study the two physical extrema in detail.

We perform the primary decomposition of $\langle f_1, f_2, f_3, f_4\rangle$ using the algorithm due to GTZ as implemented in [11]. The result contains 6 factors all of which are of dimension zero. Of these two have 1 real root, one has 2 real roots and 3 have no real roots. Of the 3 factors having real roots only the single factor with 2 roots is such that the real parts of the superfields are valued larger than 1 in the vacua. Thus the physical supersymmetric vacua of the system are given by the roots of the following ideal:

$$\langle t_1 - 5, t_2 - 5, \tau_1 - \tau_2, 9\tau_2^2 - 175\rangle \subset \langle f_1, f_2, f_3, f_4\rangle \ . \quad (27)$$

Now, for the partially F-flat component, the primary decomposition of $(((\langle \partial V, f_4, f_2 \rangle : f_3^\infty) : f_1^\infty)$ contains 3 factors all of which are again zero dimensional. Of these factors two have 2 real roots and one has no real roots at all. Of the two factors with real roots there is only one root in one of the factors for which the real parts of the superfields are both greater than 1. This is one of the two real roots of the following polynomials (the one for which t_2 is positive):

$$\langle t_1 - t_2, 21t_2^2 - 20t_2 - 900, \tau_1, \tau_2 \rangle \subset (((\langle \partial V, f_4, f_2 \rangle : f_3^\infty) : f_1^\infty) \ . \tag{28}$$

One can ask more about the properties of the above vacua, again using Sturm queries as described in Subsection 2.4 and Appendix C. We find that the non-supersymmetric vacuum described above is not a local minimum but a saddle point by testing the signs of the second derivatives of the potential. Furthermore, the vacuum in question does not obey the Breitenlohner-Freedman bound (12) and so this vacuum is not stable. Of course, in this case the resulting ideals that need to be considered have been rendered so simple by the decomposition process that one can simply find the roots of the polynomials in the prime ideals analytically. This is in fact generically the case in these flux vacua systems and is simply a consequence of the fact that prime ideals tend to take a simple form.

Solving (28) to find the position of the non-supersymmetric vacuum we obtain $t_1 = t_2 = \frac{10}{21}(1 + \sqrt{190}), \tau_1 = \tau_2 = 0$. Plotting the potential about this point we can therefore provide a check that our method is functioning correctly, as is shown in Figure 1.

We have presented this example with three main goals in mind. The first is simply to give a clear, simple example of the general discussions given in Subsection 2. The second is to demonstrate that this method is practical and powerful. We reiterate that, in the system defined by (26), we have found *all* of the isolated vacua of the system. It turns out in this case that there are three - two supersymmetric and one non-supersymmetric. To find the non-supersymmetric vacua given above and show that it and the F-flat solutions are the only such extrema present in the system using more conventional methods would be prohibitively difficult analytically. One would have to find all of the solutions to a system of 4 coupled quartics, even in this simple example. Finally, the third goal is to show that non-supersymmetric vacua of such systems do exist, even in the absence of D-terms.

4.2 Examples from String Constructions

Having whetted the reader's appetite with our toy example, we shall now delve into some systems which have been obtained in the literature in the context of string and M-theory compactifications to four dimensions. Despite the complexity of the equations which appear in these contexts, large portions of the saturation expansion (and in some cases all of it) can

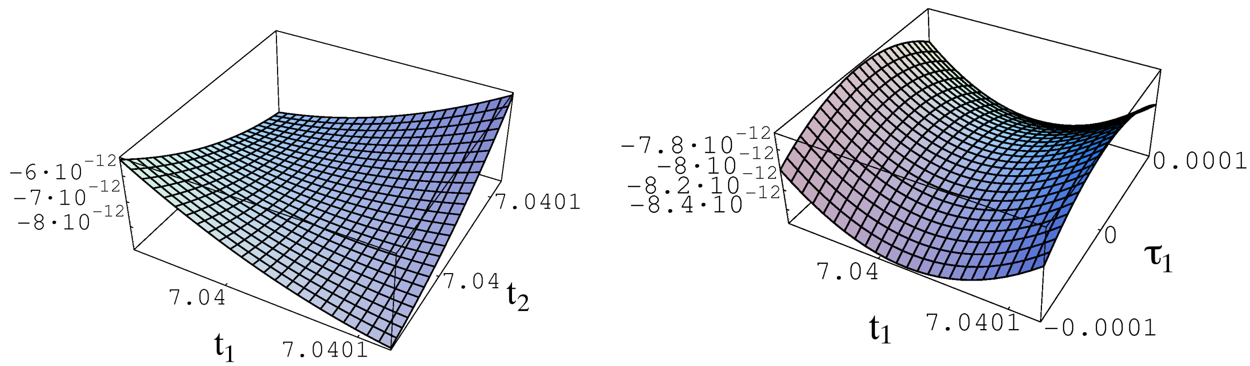

Figure 1: The non-supersymmetric vacuum for the supergravity theory specified in (26) for our toy example. The fields are $T_1 = t_1 + i\tau_1$ and $T_2 = t_2 + i\tau_2$. We have plotted the potential in two slices through field space, viz., t_1-t_2 and t_1-τ_1. A shift in the V axis of $2.09279725 \times 10^{-3}$ has been performed so that the very shallow vacuum can be plotted effectively.

still be analysed very quickly indeed. In what follows we shall first give a simple example from heterotic string theory. We shall then consider an example from M-theory where all of the moduli of the system can be stabilised perturbatively without recourse to non-geometric spaces.

4.2.1 A Heterotic Example

Let us begin with a heterotic theory compactified on one of the $SU(3)$ structure manifolds considered in [14, 15]. Of course, in a heterotic model the dilaton is unstabilised in the absence of non-perturbative effects. We shall therefore just consider the stabilisation of the analogues of the Kähler and complex structure moduli. In ignoring the dilaton in this manner the only modification to the proceeding formulae is that the $-3|W|^2$ term in equation (5) becomes $-2|W|^2$ due to a cancellation with the dilaton's F-term. For the Kähler potential and superpotential we have [14, 15]:

$$\begin{aligned} K &= -3\ln(T+\bar{T}) - 3\ln(Z+\bar{Z}) \\ W &= i(\xi + ieT) + (\epsilon + ipT)Z + \frac{i}{2}(\mu + iqT)Z^2 + \frac{1}{6}(\rho + irT)Z^3 \ , \end{aligned} \tag{29}$$

where T is the Kähler modulus and Z, the complex structure and $\xi, r, \epsilon, q, \mu, p, \rho, e$ are parameters characterising the flux and torsion on the internal space. These parameters satisfy the following constraint:

$$\xi r - \epsilon q + \mu p - \rho e = 0 \ . \tag{30}$$

As an example let us make the following parameter choices[7]:

$$\xi = -13 \ , \ r = 0 \ , \ \epsilon = -4 \ , \ q = 2 \ , \ \mu = 2 \ , \ p = 1 \ , \ \rho = 5 \ , \ e = -7 \ . \tag{31}$$

This gives rise to the following equations for the extremisation of the potential, where we have defined $T = t + i\tau$ and $Z = z + i\zeta$:

$$\begin{aligned}
0 &= 4t^2z^4 - 12\tau^2z^4 - 25z^6 + 60\tau z^4\zeta - 48\tau^2z^2\zeta^2 - 75z^4\zeta^2 + 120\tau z^2\zeta^3 - 4t^2\zeta^4 - 36\tau^2\zeta^4 \\
&\quad -75z^2\zeta^4 + 60\tau\zeta^5 - 25\zeta^6 + 24\tau z^4 + 48\tau^2z^2\zeta - 60z^4\zeta + 36\tau z^2\zeta^2 + 8t^2\zeta^3 + 72\tau^2\zeta^3 \\
&\quad -120z^2\zeta^3 + 12\tau\zeta^4 - 60\zeta^5 - 108t^2z^2 - 12\tau^2z^2 + 360tz^3 - 12z^4 + 144\tau z^2\zeta - 60t^2\zeta^2 \\
&\quad -540\tau^2\zeta^2 - 288z^2\zeta^2 + 636\tau\zeta^3 - 276\zeta^4 - 96\tau z^2 + 56t^2\zeta + 504\tau^2\zeta - 192z^2\zeta + 1152\tau\zeta^2 \\
&\quad -1068\zeta^3 - 196t^2 - 1764\tau^2 - 192z^2 + 1080\tau\zeta - 1512\zeta^2 + 6552\tau - 3744\zeta - 6084, \\
0 &= 2\tau z^4 - 5z^4\zeta + 8\tau z^2\zeta^2 - 10z^2\zeta^3 + 6\tau\zeta^4 - 5\zeta^5 - 2z^4 - 8\tau z^2\zeta - 3z^2\zeta^2 - 12\tau\zeta^3 - \zeta^4 \\
&\quad +2\tau z^2 - 12z^2\zeta + 90\tau\zeta^2 - 53\zeta^3 + 8z^2 - 84\tau\zeta - 96\zeta^2 + 294\tau - 90\zeta - 546, \\
0 &= -4t^2z^4 + 4\tau^2z^4 + 25z^6 - 20\tau z^4\zeta - 16\tau^2z^2\zeta^2 + 25z^4\zeta^2 + 40\tau z^2\zeta^3 - 12t^2\zeta^4 \\
&\quad -36\tau^2\zeta^4 - 25z^2\zeta^4 + 60\tau\zeta^5 - 25\zeta^6 - 8\tau z^4 + 16\tau^2z^2\zeta + 20z^4\zeta + 12\tau z^2\zeta^2 + 24t^2\zeta^3 \\
&\quad +72\tau^2\zeta^3 - 40z^2\zeta^3 + 12\tau\zeta^4 - 60\zeta^5 - 108t^2z^2 - 4\tau^2z^2 + 4z^4 + 48\tau z^2\zeta - 180t^2\zeta^2 \\
&\quad -540\tau^2\zeta^2 - 96z^2\zeta^2 + 636\tau\zeta^3 - 276\zeta^4 - 32\tau z^2 + 168t^2\zeta + 504\tau^2\zeta - 64z^2\zeta \\
&\quad +1152\tau\zeta^2 - 1068\zeta^3 - 588t^2 - 1764\tau^2 - 64z^2 + 1080\tau\zeta - 1512\zeta^2 + 6552\tau - 3744\zeta \\
&\quad -6084, \\
0 &= -10\tau z^4 + 16\tau^2z^2\zeta + 25z^4\zeta - 60\tau z^2\zeta^2 + 8t^2\zeta^3 + 24\tau^2\zeta^3 + 50z^2\zeta^3 - 50\tau\zeta^4 + 25\zeta^5 \\
&\quad -8\tau^2z^2 + 10z^4 - 12\tau z^2\zeta - 12t^2\zeta^2 - 36\tau^2\zeta^2 + 60z^2\zeta^2 - 8\tau\zeta^3 + 50\zeta^4 - 24\tau z^2 + 60t^2\zeta \\
&\quad +180\tau^2\zeta + 96z^2\zeta - 318\tau\zeta^2 + 184\zeta^3 - 28t^2 - 84\tau^2 + 32z^2 - 384\tau\zeta + 534\zeta^2 - 180\tau \\
&\quad +504\zeta + 624 \ .
\end{aligned} \tag{32}$$

[7]We could in principle avoid choosing parameters by working over an algebraic extension of the base field (essentially allowing polynomials with parameter coefficients), as was done in the Minkowski example in Section 3. Such a calculation would be expensive however. As such, given that the parameters in such models are quantised in any case, it is quicker to scan through a given set of values for the fluxes, and to automate the following calculations. The calculations involved here are sufficiently quick that this is a practical possibility.

The algebraic variety defined by these equations is reducible. First of all, we break up the variety according to the saturation expansion. Despite the fact that we are dealing with 4 coupled sextics in 4 variables we can calculate all but the final term (the completely non-F-flat case) in (17) for the saturation decomposition extremely quickly. The final term takes longer to complete and so we will omit it in what follows. Having obtained the various terms in the saturation expansion we go on to study each in turn.

We again use a mixture of the factorising Gröbner basis routine coupled with the GTZ primary decomposition algorithm, as implemented in [11], to break up the varieties. To find out which of the resulting zero dimensional irreducible ideals admit real roots we use the appropriate Sturm query algorithms. We also study various sign conditions evaluated on these real roots and only keep those vacua for which $\mathrm{Re}(T)$, $\mathrm{Re}(Z) > 0$. The only physical vacuum that is present is the supersymmetric vacuum which was found in [15], there are no partially F-flat vacua in this system. As such we shall move on to some more complicated cases with the aim of finding some non-supersymmetric extrema.

4.2.2 An M-Theory Example

Let us now look at another interesting example taken from M-theory. In particular we would like to consider a case where all of the moduli are perturbatively stabilised. We will return to the question of non-perturbative contributions to the superpotential in the next section.

One possibility from the literature would be type IIA string theory compactified on an orientifold of the $\frac{T^6}{Z_2 \times Z_2}$ orbifold in the presence of fluxes and torsion, as described in [32]. In particular, in their subsection 5.3, these authors provide a choice of fluxes which results in a completely stabilised supersymmetric vacuum.

If one analyses this system using the methodology we have been describing in this paper one instantly finds that $\langle \partial V \rangle$ for this system contains no zero dimensional ideals at all in its primary decomposition. In other words there are directions in field space for which this potential is completely flat. Once the presence of such a flat direction has been indicated by this formalism it is easy to spot it explicitly in the potential - in this case it corresponds to a linear combinations of some of the axions of the theory

Thus, although there is a stable supersymmetric vacuum (supersymmetric configurations of this kind automatically obey the Breitenlohner Freedman bound (12)), this system is perhaps not of such strong interest for us. For example, there is no hope of finding stable, non-supersymmetric, isolated vacua in this model. As such we shall move on to consider another possibility.

An example of the kind we would like, better suited to our purposes, is furnished by [9]. These authors consider compactifying M-theory on the coset $\frac{SU(3) \times U(1)}{U(1) \times U(1)}$. This is a manifold of $SU(3)$ structure. The resulting four dimensional supergravity theory is described by the following Kähler and superpotential [9]:

$$\begin{aligned} K &= -4\log(-i(U-\bar{U})) - \log(-i(T_1-\bar{T}_1)(T_2-\bar{T}_2)(T_3-\bar{T}_3)) \ , \\ W &= \frac{1}{\sqrt{8}}\left[4U(T_1+T_2+T_3)+2T_2T_3-T_1T_3-T_1T_2+200\right] \ . \end{aligned} \tag{33}$$

To give an idea of the complexity involved in a case such as this we note that the potential takes the form:

$$\begin{aligned} V \;=\; & \tfrac{1}{256t_1t_2t_3x^4}(40000+t_3^2\tau_1^2-400\tau_1\tau_2-4t_3^2\tau_1\tau_2+4t_3^2\tau_2^2+\tau_1^2\tau_2^2-400\tau_1\tau_3+800\tau_2\tau_3+ \\ & 2\tau_1^2\tau_2\tau_3-4\tau_1\tau_2^2\tau_3+\tau_1^2\tau_3^2-4\tau_1\tau_2\tau_3^2+4\tau_2^2\tau_3^2-24t_2t_3x^2+4t_3^2x^2-24t_1(t_2+t_3)x^2 \\ & +4\tau_1^2x^2+8\tau_1\tau_2x^2+4\tau_2^2x^2+8\tau_1\tau_3x^2+8\tau_2\tau_3x^2+4\tau_3^2x^2+1600\tau_1y-8t_3^2\tau_1y \\ & +1600\tau_2y+16t_3^2\tau_2y-8\tau_1^2\tau_2y-8\tau_1\tau_2^2y+1600\tau_3y-8\tau_1^2\tau_3y+16\tau_2^2\tau_3y-8\tau_1\tau_3^2y \\ & +16\tau_2\tau_3^2y+16t_3^2y^2+16\tau_1^2y^2+32\tau_1\tau_2y^2+16\tau_2^2y^2+32\tau_1\tau_3y^2+32\tau_2\tau_3y^2+16\tau_3^2y^2 \\ & +t_1^2(t_2^2+t_3^2+\tau_2^2+2\tau_2\tau_3+\tau_3^2+4x^2-8\tau_2y-8\tau_3y+16y^2)+t_2^2(4t_3^2+\tau_1^2-4\tau_1(\tau_3+2y) \\ & +4(\tau_3^2+x^2+4\tau_3y+4y^2)) \ , \end{aligned} \tag{34}$$

where we have defined the component fields by $T_j=-it_j+\tau_j$ for $j=1,2,3$, and $U=-ix+y$.

To obtain the equations for the extrema of this potential we must then take the derivatives of this expression with respect to all 8 fields and set them equal to zero. The result is somewhat lengthy and so we shall spare the reader the explicit full set of conditions for the extremisation of this potential. To solve these equations using normal techniques we would have to solve 8 coupled equations in 8 variables with each equation involving a quotient of a fourth order and seventh order polynomial, clearly an impossible task (even for packages such as Mathematica or Maple).

However, using our saturation and primary decomposition techniques, the problem is much more tractable. Now, in the interests of showing the diverse manners to which our methods can be applied, we will present a slightly different analysis for this system. It may be the case that one wishes to examine vacua with certain physical properties besides a specific degree of F-flatness. For example, one can ask if there are any vacua for any particular field values; for instance, say $y=0$. In terms of the variety being considered this is associated with the ideal which is generated by ∂V and the monomial y. This system, which would still be prohibitively difficult to solve with more conventional techniques, is well within the capabilities of our algorithmic techniques on a desktop computer. The search for such vacua might be physically motivated in many ways. For example, one may wish certain axions in certain models to vanish in the vacuum in order to agree with a small theta angle in a desired target theory. Since we are using the power of this formalism to look at stable, non-supersymmetric vacua, demanding such physical inputs hold true is now a reasonable thing to do.

Again, using a combination of factorising standard basis, GTZ primary decomposition and Sturm query algorithms to decompose and analyse the ideal $\langle \partial V, y \rangle$ one obtains a decomposition involving 16 factors each of which may be made up of numerous prime factors themselves. Many of the resulting prime factors are zero dimensional but only two have real roots for which the real parts of all of the superfields take values greater than 1.

As before, the prime ideals which we have extracted from the overall problem to describe these isolated loci are so simple that we can solve them explicitly to find the extrema. These turning points are described by the following ideals (the generators of which should be compared in complexity with the first derivatives of equation (34)):

$$\begin{aligned} I_1 &:= \langle 3x^2 - 100, t_1 - 2x, t_2 - x, t_3 - x, \tau_1, \tau_2, \tau_3, y \rangle, \\ I_2 &:= \langle 9x^2 - 500, 5t_1 - 2x, t_2 - x, t_3 - x, \tau_1, \tau_2, \tau_3, y \rangle \ . \end{aligned} \tag{35}$$

The simplicity of these equations shows us how useful this procedure is. In separating out the ideals that describe the isolated extrema in which we are interested from all of the rest of the turning points we have vastly simplified the discussion of the minima - in this case rendering it rather trivial. The physical root of I_1 is simply the supersymmetric vacuum of the system. This reproduces the result found in [9]. The physical root of I_2 is an isolated extremum of the system which is non-supersymmetric and anti de Sitter. These two constitute *all* of the isolated extrema of this system which obey the physical constraint we have imposed. The SUSY extremum is Breitenlohner-Freedman stable while the non-SUSY one is not.

We see that the plots of Figure 2 confirm all of the features of the non-supersymmetric extremum that our algorithmic algebro-geometric procedure rapidly predicted. We have also calculated a large part of the saturation expansion (17) for this case. We do not however find any interesting extrema beyond those described above and so shall not explicitly present this analysis here.

5 Conclusions and further work

This paper was concerned with the problem of finding vacua of four dimensional supergravities describing flux compactifications. After presenting a natural classification of such vacua we have provided two primary results within this context.

First, we have described a practical, algorithmic method for generating constraints on the flux parameters in the superpotentials of such systems. We emphasise again that these constraints can be derived as necessary conditions for the existence of *any* given kind of vacuum. In the case of supersymmetric Minkowski vacua this result is even more powerful. For these special vacua the constraints we have provided are both necessary and sufficient for the existence of such extrema.

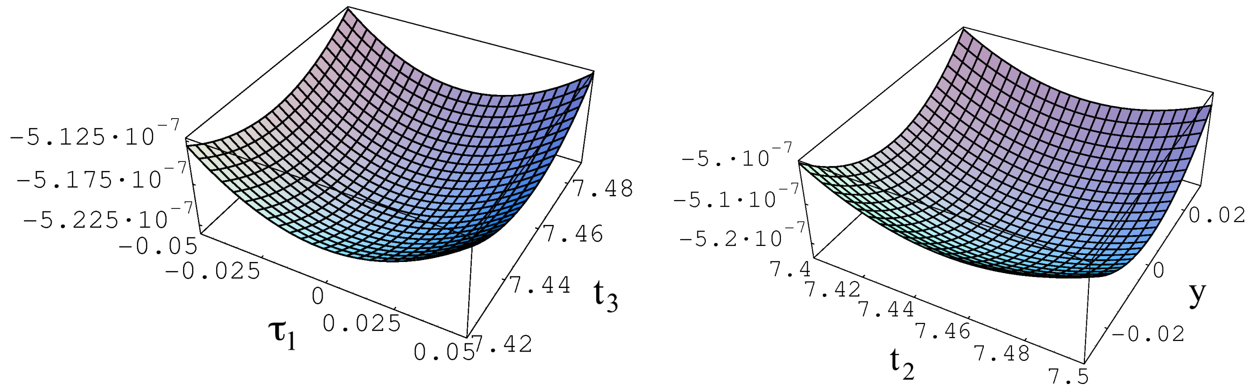

Figure 2: The non-supersymmetric extremum corresponding to the ideal I_2 in (35), for the supergravity potential specified in (34). The fields are $T_i = -it_i + \tau_i$ and $U = -ix + y$; we have here plotted the slices in (τ_1, t_3) and (t_2, y) coordinates. A shift in the V axis of 4.07×10^{-4} has been performed so that the very shallow vacuum can be plotted effectively.

Second, and perhaps more importantly, we have outlined a completely algorithmic method to find *all* of the isolated vacua of such systems, be they supersymmetric or not. In addition to the vacua themselves the methods we have described enable us to algorithmically find most of the quantities of physical interest associated with them. This includes the degree of supersymmetry they preserve, their stability as well as particle physics properties such as the Yukawa couplings in the matter sector.

What we have done is to map the extremisation problem to the language of algorithmic algebraic geometry and, in particular, of ideal theory and commutative algebra. This is not simply a hypothetical discussion. Using recent advances in computer algebra, the methods we present are powerful and allow us to solve, within seconds on an ordinary desktop computer, problems which are simply impossible with conventional techniques. We have demonstrated in concrete examples the efficiency with which our algorithms can find isolated non-supersymmetric extrema in actual systems directly derived from string and M-theoretic compactification.

One obvious extension of the work presented here would be the inclusion of non-perturbative elements in the superpotentials considered. There are several ways in which one might do this. The simplest way to proceed would be to simply introduce extra 'dummy' variables to represent any exponential functions that appear. One would then have as the desired vacuum space an algebraic variety, as described in the bulk of this paper, intersected with an exponential equation - that defining the dummy variable. This would enable one to bring the full power of algebraic geometry to bear on the difficult part of the problem. Another possibility would be to fix field values at some desired values and then solve the system to see what flux values are required to give stable vacua. In other words, we can solve for a set of the parameters rather than the fields. If this is performed carefully this will result in an algebraic variety, with the fluxes as the variables, as the object to be analysed. This approach seems to be more difficult to pursue, however, due to the quantised nature of the flux parameters in these systems.

In any event, in this paper we have restricted ourselves to perturbative superpotentials where the methods we have outlined find their simplest application. Such superpotentials can result in stabilisation of all of the moduli in geometric IIA and M-theory compactifications. Non-geometric compactifications (which give rise to a perturbative superpotential) can give rise to stabilised vacua in the other string theories as well. Perturbative vacua are interesting as they are on a somewhat firmer footing than their counterparts which rely on a mixing of perturbative and non-perturbative effects. One reason for this is that in such mixed scenarios one relies on a play off between the two types of superpotential contribution to obtain a vacuum. Although such playoffs are theoretically possible with the rest of the infinite series of non-perturbative corrections being negligible, such a situation is dependent on, for example, a very large coefficient appearing in front of the exponential terms. There is no reason to believe that such a coefficient would arise in any given model.

As a side comment we note that all of the non-supersymmetric vacua we have found thus far in any model have been partially F-flat.

Further extensions to this work are clear and numerous. As well as the inclusion of non-perturbative effects mentioned above one could consider improving the algorithms used and their application to the problem at hand. One possible such direction of improvement would be to construct a method for performing the calculations over a finite field and then separating the spurious results from the physical ones. Gröbner basis calculations over finite fields can be *much* faster than those over the rationals.

Finally, pushing these methods to their natural conclusion, one could imagine a completely automated algorithmic approach to extracting the phenomenological physics from four dimensional descriptions of string compactifications. Once a four dimensional effective theory is derived we have shown that we can scan the vacua of the system and their properties algorithmically - searching for appropriate minima with which to describe our world. Due to the complexity of these problems [33] such a program of research would have to be guided by physical insight.

Acknowledgments

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APPENDIX

A Rudiments of Computational Algebraic Geometry

Our computations throughout this paper have relied heavily upon techniques and algorithms in algebraic geometry, which may not be entirely familiar to all researchers in the field. With this appendix, which will provide a glossary on the key concepts used, we wish that the subsequent self-contained nature of this paper may serve the incipience of such methods into the study of flux vacua. Detailed exposition can be found in the texts of [26, 34], whose emphasis is on the theory, of [30], on the computation, of [10, 11, 31], on the practically implemented algorithms, as well as of [12], on a parallel application to $\mathcal{N} = 1$ gauge theories.

Algebraic Varieties and Ideals: The problem of finding the vacua of our concern, as we stated earlier, is the problem of finding the set M of simultaneous zeros of a system of polynomial equations in variables $x_1, \ldots, x_n$. Such a set M is an **affine algebraic variety**. In the language of commutative algebra, in which actual algorithms are always phrased, this set is seen as the loci of roots of an **ideal** $I(M)$ in the ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ of polynomials in x_i with coefficients in $\mathbb{C}$.

Briefly, recall that a ring is roughly a set with addition (and its inverse, subtraction) and multiplication, but no division. Indeed, the sum, difference and product of two polynomials remain a polynomial while the ratio does not. An ideal is a subset, which, when multiplied by any element, remain in the subset. To intimate the relation between the algebraic object, viz., the ideal I and the geometric object, viz., the variety L, the standard notation is to use $I(L)$ and $L(I)$ when they correspond.

To be explicit, we use $\langle f_1, \ldots, f_k \rangle$ to denote the ideal of generated by the polynomials f_i, i.e.,

$$I = \langle f_1, \ldots, f_k \rangle = \left\{ \sum_{i=1}^{k} h_i(x_1, \ldots, x_n) f_i \right\} \subset \mathbb{C}[x_1, \ldots, x_n] \tag{36}$$

for polynomials h_i. In this notation, addition and multiplication between two ideals is easily defined as the addition and multiplication of all combinations of the generators. Quotients will be defined shortly.

Radical Ideals: Next, $I(M)$ can contain more information than is physically needed. Multiplicities in the roots describe the same set of points. Recall the example in the text: $x = 0$ and $x^2 = 0$ describe the same set of points even though $\langle x \rangle$ and $\langle x^2 \rangle$ are two different ideals. This ambiguity is resolved by defining the radical $\sqrt{I}$ of the ideal of I in a ring R:

$$\sqrt{I} := \{r \in R | r^n \in I \text{ for some } n \in \mathbb{Z}_+\} \ . \tag{37}$$

The Hilbert Nullstellensatz then states that, for any ideal J, the ideal $I(L(J))$ corresponding to the variety $L(J)$ whose points are determined by J is equal to the radical ideal $\sqrt{J}$. In other words, the radical ideal is the "minimal" ideal corresponding to the variety M which drops all the redundant information on the multiplicities of the zeros. Thus, we can refine to the study of the radical ideal $\sqrt{I(M)}$ corresponding to our zero-set M. Popular algorithms which perform this step can be found in [17] and are implemented in [10, 11].

Primary Decomposition: The radical ideal $\sqrt{I(M)}$ may still be reducible in the sense that the variety $xy = 0$, for example, clearly consists of two irreducible components $x = 0$ and $y = 0$. To obtain the elemental constituents of $\sqrt{I(M)}$ we must then decompose it into **prime ideals**, ideals p for which (just like a prime number), $ab \in p$ implies that $a \in p$ or $b \in p$. Such a process is called primary decomposition[8]. The theory was originally due to Lasker-Noether, with the first algorithm by Hermann. Today, it constitutes one of the most exciting areas of research in computer algebra, with popular algorithms by Shimoyana-Yokoyama, Eisenbud-Huneke-Vasconcelos, and Gianni-Trager-Zacharias as implemented in [10, 11]. We shall describe the last of these algorithms, which we have used throughout this paper, in some detail in Appendix B. We therefore have the decomposition of $\sqrt{I(M)}$ as the finite intersection of prime ideals $P(i)$, i.e., $\sqrt{I(M)} = \cap_i P(i)$.

Real Roots: After decomposing into irreducible components, one can then compute the dimension (corresponding to the number of flat directions) of each piece $P(i)$. A method for checking whether an ideal is zero dimensional, for example, is briefly described in appendix C. In the case that the ideal $P(i)$ is zero-dimensional, the component corresponds to no more than a (discrete) set S_i of points. Physically, this means that this component of the vacuum has been completely isolated. One could determine the cardinality of S_i (the number of roots of the polynomial system); this is called the **virtual dimension** of the zero-dimensional ideal $P(i)$. In particular, we are interested in the set of real roots, which is a special subset of S_i.

[8]Strictly, irreducible varieties correspond to **primary ideals** which are ideals I for which $ab \in p$ implies that $a \in p$ or $b^n \in p$ for some integer n, a weaker condition than primality. However, since radicals of primary ideals are prime and we are already starting with a radical ideal, it suffices to study the stronger condition of prime decomposition.

Algorithms have been developed to deal with real roots [21]. We shall discuss this further in appendix C.

Quotients and Saturations of Ideals: During the course of our analysis we need the concepts of saturations and quotients of ideals. An ideal quotient of an ideal $I \subset R$ with respect to $f \in R$ is simply defined as follows:

$$(I : f) := \{g \in R | gf \in I\} \ . \tag{38}$$

In general, the quotient $(I : J)$ of an ideal I by an ideal J is the set of elements $g \in R$ such that $g \cdot J$ is contained in I. The definition of a saturation of an ideal is then a simple extension of this idea:

$$(I : f^\infty) := \{g \in R | gf^N \in I, \text{for some } N \in \mathbb{Z}_{>0}\} = \bigcup_{n=1}^{\infty} I : f^n \ . \tag{39}$$

The second equality is important and is the origin of the infinity in the notation: saturation quotients out all powers of f. Geometrically, this means that $L(I : f^\infty)$ corresponds to the subvariety of $L(I)$ for which $f \neq 0$.

Quotient Rings: The last concept that we shall require, for use in later appendices, is that of a quotient ring. For an ideal I in a ring R the quotient ring R/I is simply defined to be the set of all elements in R where two elements are regarded as equivalent if their difference is an element in I. Physically the quotient ring corresponds to the set of all polynomial functions where two functions are only regarded as different when they take different values on the locus $L(I)$ which is defined by the ideal I.

Gröbner Basis: The first step in almost all algorithms in computational algebraic geometry is to place the generators of the ideal of multi-variate polynomials into a so-called Gröbner Basis. This is a generalisation of Gaussian elimination for a multivariate linear system to general polynomials.

In computational algebraic geometry, the Gröbner basis is determined by (modifications and improvements of) Buchberger's algorithm (see for example [31]). The Buchberger algorithm proceeds as follows. Start with an ideal I.

1. Set $\mathcal{G} =$ generators(I).

2. For any pair of polynomials A, $B \in \mathcal{G}$ form the S polynomial (described below).

3. Reduce the S polynomial with respect to $\mathcal{G}$.

4. If the reduction is non-zero add the result to $\mathcal{G}$.
5. Repeat from step 2 until all pairs of polynomials in $\mathcal{G}$ give S polynomials which reduce to zero. $\mathcal{G}$ is then the Gröbner basis.

In the above one needs to understand the process of reduction and what an S polynomial is. Both of these concepts rely on the introduction of **monomial orderings**. An ordering $>$ is simply a rule which allows us to unambiguously compare any two monomials in the variables and say which one is higher in a list of all monomials. For example the Lexicographic ordering with respect to the variables a, b, c just says that monomials are ordered, firstly according to the power of a they contain (highest first), then according to the power of b and finally that of c. So, for example, a^2bc would be ordered higher than ab^2c^4.

The reduction process of polynomial A relative to polynomial C is then simply as follows. We subtract some (possibly monomial) multiple of C from A in such a manner as to cancel A's leading term with respect to the ordering $>$. If the leading term can not be canceled in this way A is simply left alone.

The S polynomial of two polynomials A and B is simply given as follows. Multiply A and B by the lowest degree monomials possible so that the leading terms of the two results, A' and B', become equal. One then simply subtracts one from the other, so that the leading terms cancel: $S = A' - B'$.

Gröbner bases have many uses, some of which we shall encounter later in these appendices. One particularly useful feature of these sets of polynomials is that the reduction of any polynomial with respect to $\mathcal{G}$ does not depend upon the order in which we use the polynomials therein in the reduction procedure. Another vital property is that given a monomial ordering, the Gröbner basis (reduced with respect to itself) is unique for any given ideal. Unfortunately, one of the biggest hurdles in computational algebraic geometry is that the algorithm for determining the Gröbner basis can be very intensive.

B Primary Decomposition Algorithms

In this section, we discuss in a little more detail the key algorithm used throughout the paper. There are now several primary decomposition routines available [18, 19, 20], many of which are implemented in algebra systems such as [10, 11]. We make extensive use of the algorithm due to Gianni, Trager, and Zacharias (GTZ) [18] in this paper and so we shall now give a brief description of the basics of the algorithm's workings, following closely such texts as [17, 31].

The GTZ algorithm is built around the same splitting principle as was used in Subsection 2.5; that is, if $(I : f^\infty) = (I : f^l)$ for some l, then

$$I = (I : f^\infty) \cap \langle I, f^l \rangle \ . \tag{40}$$

Given this fact the GTZ algorithm works by specifying the polynomials f and by reducing the primary decomposition of an ideal of dimension d to a problem involving primary decompositions of zero dimensional ideals. An existing algorithm can then be employed to primary decompose the zero dimensional ideals. We thus split our description into these two halves. First, we describe how the GTZ algorithm reduces everything to zero dimensional primary decompositions and finds a suitable f. Second, we give a brief discussion of how one obtains a primary decomposition of a zero dimensional ideal.

B.1 GTZ reduction

The first step is to reduce the d dimensional decomposition problem to a 0 dimensional one. We start with an ideal I in the ring $\mathbb{C}[X_1, \ldots, X_n]$. First, choose a maximal subset $Y = \{Y_1, ..., Y_d\}$ of the variables of the ring, $X = \{X_1, \ldots, X_n\}$, such that these variables are independent mod I. That is, $I \cap \mathbb{C}[Y_1, ..., Y_d] = \{0\}$. Geometrically, Y are the variables along $L(I)$ and $X \setminus Y$, transverse. Thus, d is the dimension of I. Now take the polynomials defining I to be polynomials in $I_{\mathbb{C}(Y)[X\setminus Y]} \subset \mathbb{C}(Y)[X \setminus Y]$. That is, pretend that the Y variables are coefficients. The ideal $I_{\mathbb{C}(Y)[X\setminus Y]}$, with all Y variables in I considered as coefficients, is then zero dimensional.

Now, for our original ring $\mathbb{C}[X]$, choose a monomial ordering $<$, with $Y_i < X_j$ for all i whenever $X_j \in (X \setminus Y)$. Take a Gröbner basis $\mathcal{G}$ of I with respect to $<$. This is then also a Gröbner basis of $I_{\mathbb{C}(Y)[X\setminus Y]}$, via restriction of $<$ to $X \setminus Y$. We are now in a position to isolate the f which GTZ employ. We take f to be the least common multiple of the leading coefficients of the polynomials in $\mathcal{G}$, with these polynomials taken to lie in $\mathbb{C}(Y)[X \setminus Y]$. The crucial observation is then the following:

$$I_{\mathbb{C}(Y)[X\setminus Y]} \cap \mathbb{C}[X] = (I : f^\infty) \ . \tag{41}$$

Thus, of the two halves of the saturation decomposition $I = (I : f^\infty) \cap \langle I, f^l \rangle$, the first factor can be addressed by a zero-dimensional primary decomposition (to which we turn in the next subsection), leaving us with only $I' = \langle I, f^l \rangle$, to deal with. We can then repeat the above process on I', and iterate until when there is nothing new in the second factor, i.e., when a factor we already have lies within the starting point for the next iteration.

B.2 Zero dimensional Primary Decomposition

Finding a primary decomposition of a zero dimensional ideal is relatively straightforward using Gröbner bases. Any zero dimensional ideal I can be put in a so-called "general position" with respect to the lexicographical ordering induced from $X_1 > ... > X_n$. This is defined by the following properties:

- The primes $P(i)$ in the primary decomposition of I have a reduced Gröbner basis with respect to the same ordering of the form

$$\{P(i)\} = \{X_1 - h_1(X_n),\ \ldots,\ X_{n-1} - h_{n-1}(X_n),\ h_n(X_n)\}\ . \tag{42}$$

  Here, we have $h_i \in \mathbb{C}[X_n]$, i.e., they are simply polynomials in X_n.

- The ideals $P(i)$ are coprime. In other words, the polynomials h_i have as their greatest common divisors just an element of the coefficient field $\mathbb{C}$, viz., a constant.

Write $\mathcal{G}$ for a corresponding minimal Gröbner basis and define $\{h\} = \mathcal{G} \cap \mathbb{C}[X_n]$. There is then a theorem [17] which states that if $h = h_1^{l_1} ... h_f^{l_f}$ is the factorisation of h into a product of powers of pairwise non-associated irreducible factors, then the primary decomposition is just given by:

$$I = \bigcap_{j=1}^{f} \langle I, h_j^{l_j} \rangle\ . \tag{43}$$

An example of how this theorem can be used to implement an appropriate algorithm can be found in [17], as can various details.

C Sturm Queries and Real Roots

One of the topics of primary importance within this paper is the discussion of finding real roots of zero dimensional ideals. We shall thus briefly describe some of the mathematical ideas involved in this appendix, following closely the excellent treatments of [21, 22].

To commence, a finite set of polynomials within $\mathbb{C}[X_1, ..., X_k]$ is zero dimensional iff any Gröbner basis of the associated ideal contains a polynomial with leading monomial $X_i^{d_i}$ for each $i \in [1, k]$. Once a zero dimensional system has been identified one of the central notions in the study of its real roots is that of a **Sturm query**. Let $P \in \mathbb{R}[x]$ be a real polynomial and Z, a set of points. The Sturm query is given by the following expression:

$$SQ(P, Z) = \sharp\{x \in Z | P(x) > 0\} - \sharp\{x \in Z | P(x) < 0\}\ . \tag{44}$$

If we had this function, then, for a zero-dimensional ideal I of real polynomials, the number of real roots is simply $SQ(1, r(I))$, where $r(I)$ is the (discrete) set of real roots for I.

Moreover, we can also test sign conditions, another real algebro-geometric device which we use extensively in the paper. In such calculations we wish to know the sign taken by a given polynomial P evaluated at the elements of $r(I)$. We note that, by definition, the following system of equations holds:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sharp\{x \in r(I) | P = 0)\} \\ \sharp\{x \in r(I) | P > 0)\} \\ \sharp\{x \in r(I) | P < 0)\} \end{pmatrix} = \begin{pmatrix} SQ(1, r(I)) \\ SQ(P, r(I)) \\ SQ(P^2, r(I)) \end{pmatrix} . \tag{45}$$

Once the Sturm queries are known, we can immediate solve for the quantities $\sharp\{x \in r(I) | P = 0, P > 0, \text{ or } P < 0\}$, which are what we are after. One can also, in the same way, ask about the signs of lists of polynomials. This just involves the study of a bigger matrix equation.

Thus we see that, once we know how to algorithmically compute Sturm queries, we can find the number of real roots of an ideal as well as the signs various polynomials take on those roots. How then is a Sturm query obtained algorithmically? The starting point here is to notice that if I is zero dimensional then the quotient ring $R_Q = R[X_1, \ldots, X_n]/I$ is a finite-dimensional R-vector space A. We can imagine taking a basis consisting of functions which are 1 on one root and zero on all the others, with one such function in the basis for each root. One can then obtain any function on the roots by combining multiples of these basis elements in the correct manner. We can define various linear maps on this space. One such map, $L_f : A \mapsto A$ can just be defined to be multiplication within R_Q by a function f. One can also consider bilinear maps $H_g : A \times A \mapsto R$ defined by $H_g(f_1, f_2) = \text{Trace}(L_{f_1 f_2 g})$. Clearly the matrix associated to H_g in some basis for A is symmetric.

A theorem due to Hermite states that the Sturm query $SQ(g, r(I))$ is simply given by the signature of this symmetric matrix. This is, in fact, intuitively obvious when thinking in terms of the basis described above. This matrix can be obtained algorithmically using Gröbner bases [21]. Algorithmically the signature of symmetric matrices is easy to find. All of the eigenvalues of a symmetric matrix are real and are given by the roots of its characteristic polynomial. The number of positive roots is then determined by essentially Descartes' law of signs (or its generalisation, the Budan-Fourier theorem) [21], i.e., by examining the signs of the coefficients of the characteristic polynomial.

The methods describe above are not necessarily the fastest way to obtain the results required, particularly the number of real roots [21, 22]. They are however the simplest to understand. The reader interested in further details of these kinds of calculations is referred to [21, 22]. From a practical stand point, all of the algorithms concerned with real roots which we require have been implemented in [11] by Tobis [22].

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