MIT-CTP 3761

CMSZJU-06005

Counting BPS Operators in Gauge Theories

- Quivers, Syzygies and Plethystics

Sergio Benvenuti[1,2*], **Bo Feng**[3,4†] **Amihay Hanany**[1‡], **Yang-Hui He**[5,6§],

[1] *Center for Theoretical Physics, M.I.T., Cambridge, MA02139, U.S.A.*

[2] *Joseph Henry Laoratories., Princeton University, Princeton, NJ., U.S.A.*

[3] *Blackett Lab. & Inst. for Math. Science, Imperial College, London, SW7 2AZ, U.K.*

[4] *Center of Mathematical Science, Zhejiang University, Hangzhou 310027, P. R. China*

[5] *Merton College, Oxford University, Merton Street, Oxford, OX1 4JD, U.K.*

[6] *Mathematical Institute, Oxford University, 24-29 St. Giles', Oxford, OX1 3LB, U.K.*

Abstract

We develop a systematic and efficient method of counting single-trace and multi-trace BPS operators with two supercharges, for world-volume gauge theories of N D-brane probes for both $N \to \infty$ and finite N. The techniques are applicable to generic singularities, orbifold, toric, non-toric, complete intersections, et cetera, even to geometries whose precise field theory duals are not yet known. The so-called "Plethystic Exponential" provides a simple bridge between (1) the defining equation of the Calabi-Yau, (2) the generating function of single-trace BPS operators and (3) the generating function of multi-trace operators. Mathematically, fascinating and intricate inter-relations between gauge theory, algebraic geometry, combinatorics and number theory exhibit themselves in the form of plethystics and syzygies.

* `s.benvenuti@sns.it`

† `b.feng@imperial.ac.uk`

‡ `hanany@mit.edu`

§ `hey@maths.ox.ac.uk`

Contents

1 Introduction

The study of BPS states in a quantum field theory is of unquestionable importance. The purpose of this note is to discuss the set of all mesonic BPS gauge invariant operators (GIO) with two supercharges which appear in the chiral ring of a generic $\mathcal{N} = 1$ supersymmetric gauge theory that lives on a D3-brane which probes a singular Calabi Yau (CY) manifold[1]. For arbitrary singularities, finding the gauge theory living on the D3-brane is intricate. The simplest class is the orbifolds, the study of which began with [2, 3, 4, 5, 6, 7, 8]. The next simplest class is the toric singularities, the investigation of which was initiated by [8, 9, 10, 11]. Interesting duality structures of these theories have been expounded in [12, 13, 14, 15]. It is recently realised that the toric theories are, in fact, best described using a bi-partite periodic tiling of the two dimensional plane, a so-called "dimer model" [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] (also cf. [29, 30] for recent mathematical treatments).

When the manifold is non-orbifold and non-toric there is no current systematic way of describing the gauge theory even though some examples exist in the literature. For example, the higher del Pezzo series [31] and certain deformations of toric singularities [27] have been addressed. In this paper we will see how one can describe the single- and multi-trace operators in terms of generating functions which can be computed for both

[1] Some preliminary results were annouced in [1].

toric and non-toric manifolds. In fact, the computations we will see can calculate the generating functions even for cases in which the gauge theory is not precisely known - either the superpotential is missing or even the quiver itself is not known.

The discussion on GIO's in the chiral ring can be divided into few parts as follows. Given a gauge theory description of the theory on the D-brane, there are several problems of interest:

Global $U(1)$ Charges: One would like first to identify the set of global $U(1)$ charges of this theory. One charge out of this set is singled out to be the **R-charge** and the other charges can be generically called global non-R charges. The most useful way of thinking about these charges is by introducing the holographically dual gravity description. A set of D3-branes on a singular conical CY is holographically dual to an $AdS_5 \times Y_5$ background where Y_5 is a Sasaki Einstein (SE) manifold (cf. [32] and references therein). The global charges of the gauge theory are dual to gauge fields in AdS_5. These gauge fields can be divided into two sets – one set originates from the Type IIB metric, those are typically referred to as the isometry of the SE manifold, and the other set comes from the Type IIB 4-form. The R-charge is always part of the isometry group of the SE manifold. The traditional name which was given to the charges coming from the metric are **flavor charges** and those which come from the 4-form are called **baryonic charges**.

The isometry group of the SE manifold has a maximum rank of 3, in which case the SE manifold and its CY cone are called *toric*; the minimum rank is 1, in which case the corresponding $U(1)$ charge is precisely the R-charge. The number of baryonic charges is in principle unbounded and is given by the third homology of the SE manifold. Most cases which were studied in the literature have one baryonic symmetry, the prototypical example being the conifold [33]. Currently there are extensive studies of cases with more than one baryonic charge, the simplest being the Suspended Pinch Point (SPP) [9, 18, 34], as well as the more complicated $X^{p,q}$ family [50].

Counting Gauge Invariant Operators: Given the set of $U(1)$ symmetries, R, flavor, baryonic, etc., say, n of them, each gauge invariant operator in the chiral ring carries a set of charges under these symmetries. We will assign a generic complex variable $t_i, i = 1 \dots n$ to each such charge and define a function $f(\{t_i\})$ to be the **generating function** of all these operators. This function f has, by definition, an expansion in terms of monomials in $\{t_i\}$ such that the coefficient, $c_{k_1,\dots,k_n}$ of $t_1^{k_1} \cdots t_n^{k_n}$ is integer and counts the number of operators of charges $(k_1, \ldots, k_n)$,

$$f(\{t_i\}) = \sum_{i_1,\ldots,i_k} c_{k_1,\ldots,k_n} t_1^{k_1} \ldots t_n^{k_n} \ . \tag{1.1}$$

Our goal is to compute such functions for a multitude of cases.

Our ultimate wish is that for any CY manifold we would like to know

1. The set of single-trace BPS operators, the generating function is denoted by f;

2. The set of multi-trace BPS operators, the generating function is denoted by g;

3. For N D3-branes at the singular CY we would like to know the dependence on N. Namely, we would like to know how many independent single-trace and multi-trace operators are there in the chiral ring for a given set of charges. For a finite N this turns out to be a much more difficult task since there are matrix relations for a finite size matrix that need to be taken into account. Nevertheless, we propose a nice solution to this as well; the generating functions in this case will be denoted as f_N for single-trace and g_N for multi-trace.

As will be discussed in detail in later sections there is an important function which beautifully relates the single-trace generating function and the multi-trace. Namely, g_N can be simply computed from f_N using the so-called *"Plethystic Exponential."* This function has been used in physics several times in the past and we believe it should go into the literature more often as it plays a crucial role in counting problems such as the one dealt with here[2]. Conversely one can use the so-called *"Plethystic Logarithm"* which is the inverse function to the plethystic exponential and computes f_N from g_N. The ability to switch between f and g will turn out to be a key tool in analyzing the theories we are interested in and to reveal new pieces of information which were either previously unknown or alternatively not well discussed.

Having presented a host of functions and concepts, it would be most expedient to exemplify them in a context with which the readers are well-acquainted. We shall do so for the famous D3-brane theory on $\mathbb{C}^3$ in the next section. Having whetted the readers' appetites, the plan for the remainder of the paper is as follows. We begin with the large N limit and present the solution to questions (1) and (2) above. In §3, we show how to construct f, the generating function for single-trace GIO's. This is

[2] A. H. would like to thank Marcos Mariño for demonstrating the properties of this function [53].

a Hilbert-Poincaré counting problem. We exemplify with orbifolds, toric varieties and the del Pezzo family. We take an interlude in §4 and examine this counting problem using the graphical perspectives of dimers. Then, in §5, we construct the generating function g, which count the multi-trace GIO's. The relationship between f and g will turn out to be a plethystic one. In due course, we will show how plethystics actually encode not only the GIO's counting, but also the defining equation of the singularity. Interesting partition identities as well as syzygies in graded polynomial rings emerge. Having constructed the generating functions, we then calculate the asymptotic behaviour thereof in §6, using results from combinatorics and analytic number theory. Finally, we use the above formalism to address the more difficult problem of finite N in §7 and show how plethstics again solves the counting problem and how they encode the geometry. We conclude with perspectives in §8.

2 $\mathbb{C}^3$: An Illustrative Example

As promised in the introduction, we begin with a familiar example to illustrate the various generating functions. Here, the computation can be done without using the more general techniques which will follow in the rest of the paper. This example is of course for the archetypal example of the AdS/CFT correspondence, the case in which the CY manifold is trivially $\mathbb{C}^3$ and its associated SE manifold, S^5 [35]. There are no baryonic charges in this case since the third homology of S^5 is trivial and the isometry group is $SU(4)$ with rank 3, meaning that this CY manifold is actually toric and the number of $U(1)$ charges is 3. We can thus define 3 corresponding variables, t_1, t_2, t_3, which will then measure these three $U(1)$ charges in their powers, as explained above. The gauge theory is the $\mathcal{N} = 4$ gauge theory with $U(N)$ gauge group which in $\mathcal{N} = 1$ language has 3 adjoint chiral multiplets which we will denote as x, y and z. Being toric, this CY manifold admits a description in terms of periodic bi-partite tilings of the two dimensional plane and in fact is given by the simplest of them all - tilings by regular hexagons [17].

We are interested in operators in the chiral ring and therefore we need to impose the F-term relations coming from the superpotential $W = \mathrm{Tr}(x[y, z])$. The F-terms hence take a particularly simple form: $[x, y] = [y, z] = [z, x] = 0$, i.e., all chiral adjoint fields commute. The generic single-trace GIO in the chiral ring will then take the form of $\mathrm{Tr}(x^i y^j z^k)$. It is then natural to assign t_1 as counting the number of x fields, t_2, the number of y fields and t_3, the number of z fields in a GIO. There will therefore be a corresponding monomial $t_1^i t_2^j t_3^k$ for each gauge invariant of charges i, j, and k, respectively. In fact, there will be precisely one for each triple of charges, provided each of i, j, and k are non-negative. Putting all of this together, we find that the generating function f takes the form

$$f(t_1, t_2, t_3; \mathbb{C}^3) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} t_1^i t_2^j t_3^k = \frac{1}{(1-t_1)(1-t_2)(1-t_3)} \ . \tag{2.1}$$

To be more precise, in the above form we did not take into account any relations that a finite matrix should satisfy, therefore, as mentioned earlier, this result is strictly valid for the case of $N = \infty$. Therefore, using the notation introduced above we should write

$$f_\infty(t_1, t_2, t_3; \mathbb{C}^3) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}. \tag{2.2}$$

A General Feature for Toric CY: Note that in eq. (2.2) the coefficients c_{ijk} appearing in the general expansion

$$f_\infty(t_1, t_2, t_3) = \sum_{ijk} c_{ijk} t_1^i t_2^j t_3^k \tag{2.3}$$

are all equal to either 1 or 0. This means that for a given set of charges, i, j, k, there is either one operator carrying these charges or not, but there can not be more than one. Indeed this is a generic feature which is obeyed for every toric singular CY. More explicitly there is a one-to-one correspondence between single-trace GIOs and integer lattice points in the dual cone of toric diagram [10, 27, 37]. This property is reminiscent of some kind of a fermionic degree of freedom that carries this set of charges. In contrast, for the non-toric case, it is shown in [27] that there are, in general, multiple-to-one mappings between single-trace GIOs and given charges. The reason is clear. In the toric case, we have two extra $U(1)$ flavor symmetries besides the R-symmetry, which is big enough to distinguish finely, while for non-toric case we do not have these extra symmetries.

Let us look at the set of charges i, j, k for which c_{ijk} are not zero. They form a sub-lattice of the three dimensional lattice which has the form of a cone. Indeed, this sub-lattice is the so-called "positive octant" for which $i \geq 0, j \geq 0, k \geq 0$. This feature of a cone structure will also be general for every CY manifold, the form of this cone is interesting and will be discussed in detail in §4. One can think of the function f_∞ as a theta-function over the lattice points of the cone and is a characteristic function of this cone. It is worth to notice that from results in [27], it seems that there is a lattice structure for both toric and non-toric cases. The difference is that for the toric case the lattice is 3-dimensional while for non-toric case the dimension is lower.

If on the other hand we are interested in counting the number of BPS operators which carry a given fixed scaling dimension, say $\mathrm{Tr}(x^i y^j z^k)$ of dimension $i+j+k = \frac{3}{2}R$, we need to set $t_1 = t_2 = t_3 = t$ in eq. (2.2) and get the generating function for all operators. In other words, we have to forget the other two $U(1)$ flavor symmetries and use the fact that all variables x, y, z have same R-charge $\frac{2}{3}$. Hence,

$$f_\infty(t;\ \mathbb{C}^3) = \frac{1}{(1-t)^3} = \sum_{m=0}^{\infty} \binom{m+2}{2} t^m \tag{2.4}$$

and the number of GIO's of given R-charge $R = \frac{2}{3}m$ is $\binom{m+2}{2}$, corresponding to the completely symmetric rank m representation of $SU(3)$ that acts on x, y, and z in the fundamental representation.

Single-Trace and Multi-Trace at $N \to \infty$: Having studied f_∞, let us now look at the function f_1, generating the single-trace operators for the case of one D3-brane on $\mathbb{C}^3$. Clearly, the adjoint fields x, y, and z are complex variables and not matrices and therefore any product of two or more of these matrices is a multi-trace operator. As a result, there are only 4 single-trace operators in this case: the identity operator, x, y, and z. We can therefore use their representation in terms of $t_i, i = 1, 2, 3$, sum them and write:

$$f_1(t_1, t_2, t_3) = 1 + t_1 + t_2 + t_3. \tag{2.5}$$

Next, we notice an interesting relation between g_1 and f_∞. Let us look at the set of operators of the form $\mathrm{Tr}(x^i y^j z^k)$ for the case in which the number of D3-branes is $N \to \infty$. Each such operator is represented by the monomial $t_1^i t_2^j t_3^k$ and can be thought of as a multi-trace operator for the case of the number of D3-branes being $N = 1$. This implies that g_1, the generating function for multi-trace operators for one D3-brane is equal to f_∞, the generating function for single-trace operators for infinitely many D3 branes,

$$g_1 = f_\infty\ . \tag{2.6}$$

Can we now find some functional dependence between f_1 and g_1? Combining expressions eq. (2.6), eq. (2.5) and eq. (2.2), we have

$$\begin{aligned} g_1(t_1,t_2,t_3) &= \frac{1}{(1-t_1)(1-t_2)(1-t_3)} = \exp[-\log(1-t_1)-\log(1-t_2)-\log(1-t_3)] \\ &= \exp\left(\sum_{r=1}^{\infty}\frac{t_1^r+t_2^r+t_3^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty}\frac{f_1(t_1^r,t_2^r,t_3^r)-1}{r}\right). \end{aligned} \tag{2.7}$$

The last relation

$$g_1(t_1,t_2,t_3) = \exp\left(\sum_{r=1}^{\infty}\frac{f_1(t_1^r,t_2^r,t_3^r)-1}{r}\right) = f_\infty(t_1,t_2,t_3) \tag{2.8}$$

turns out to be a key relation and is satisfied for any CY manifold, toric or otherwise. The function g_1 is then said to be the **Plethystic Exponential** of f_1. This relation in fact generalizes to any N and we find that g_N is the plethystic exponential of f_N. We will discuss this extensively in §5 and §7.

We are now ready to write down the expression for the generating function g_∞ of multi-trace BPS GIO's in the chiral ring in the $\mathcal{N}=4$ theory, corresponding to $N\to\infty$ D3-branes on $\mathbb{C}^3$. It is again, the plethystic exponential, this time of f_∞ in eq. (2.2):

$$g_\infty(t_1,t_2,t_3) = \exp\left(\sum_{r=1}^{\infty}\frac{f_\infty(t_1^r,t_2^r,t_3^r)-1}{r}\right) = \exp\left(\sum_{r=1}^{\infty}\frac{\frac{1}{(1-t_1^r)(1-t_2^r)(1-t_3^r)}-1}{r}\right). \tag{2.9}$$

Note that g_∞ has an expansion

$$g_\infty(t_1,t_2,t_3) = \sum_{ijk} d_{ijk}t_1^i t_2^j t_3^k, \tag{2.10}$$

where the coefficients d_{ijk} are non-zero precisely when the coefficients c_{ijk} of f_∞ are non-zero. However, while c_{ijk} can be at most 1, d_{ijk} has a very fast growth and in fact grows exponentially. It is therefore a problem of interest to find what is the large charge behavior of d_{ijk}. We see that the multiplicity of BPS states for fixed R charge, $R=\frac{2}{3}k$, is

$$g_\infty(t,t,t) = \exp\left(\sum_{r=1}^{\infty}\frac{\frac{1}{(1-t^r)^3}-1}{r}\right) = \sum_{k=0}^{\infty} d_k t^k\ . \tag{2.11}$$

We will present in §6 detailed discussions of how to obtain d_k for large k.

Single-Trace and Multi-Trace at Finite N: For finite N, the situation is in general much more involved. Nevertheless, the multi-trace result g_N can be obtained from f_N by plethystics. We will present the systematic treatment for arbitrary singularities in §7.

3 Counting Gauge Invariants: Poincaré Series and Single-Trace

Having stated our problem and enticed the reader with the example of $\mathbb{C}^3$, we are now ready to attack the general CY singularity. Our strategy will be to first examine the simpler case of $N \to \infty$ and then the more involved case of finite N.

Beginning with the large N situation, we first find the generating function f for the single-trace GIO's. Then, in §5, we will show how the plethystic exponential (PE), extracts g, the generating function for the multi-trace GIO's, from f. Indeed, because the multi-trace GIO's are composed of products of the single-trace ones, PE is expected to be a version of counting integer-partitions. We would like to emphasize that the counting automatically encodes more than merely the matter content, but, furtively, the superpotential as well. In other words, we will be concerned with a true counting of the GIO's with the F-term constraints automatically built in. We will check in all examples below that this is indeed so by showing that the moduli space is explicitly the CY 3-fold, as is required in D-brane probe theories.

How, then, do we compute f given the geometrical data of the CY? It turns out that we could appeal to some known methods in mathematics. In projective algebraic geometry, an important problem is to count the number of generators of graded pieces of polynomial rings, the generating functions of this type are called **Hilbert-Poincaré series**.

We shall borrow this terminology and refer to the function f for the single-trace GIO's as the Poincaré series for the associated $\mathcal{N} = 1$ gauge theory; it shall soon be seen that this appropriation is a conducive one. In this section, we proceed stepwise along the various known classes of CY singularities which the D3-brane can probe. We start with orbifolds and see the Poincaré series in the mathematical sense is precisely what is needed. Next, we address toric CY singularities; here, using the techniques of (p, q)-webs and 2-dimensional tilings (dimers), we construct f from the toric diagram. Then, we proceed to the del Pezzo family of singularities.

3.1 Orbifolds and Molien Series

Given a finite group, it is a classical problem to find the generators of the ring of polynomial invariants under the group action. The theory matured under E. Nöther and T. Molien (cf. e.g. [38]). In our quiver gauge theory, the single-trace GIO's are polynomial combinations of fields which are invariant under the group action. Because we are assuming large N, no extra relations arise beside these from the F-terms, and the problem of computing f reduces to simply counting the number of algebraically independent polynomials one could construct of degree n that are invariant under the group. The problem is a mathematical one and was solved by Molien; the Poincaré series is named **Molien series** in his honour.

Let us be concrete and specialise to the orbifolds of our concern, viz., 3-dimensional CY orbifolds $\mathbb{C}^3/G$, with G a discrete finite subgroup of $SU(3)$. Such singularities were first classified by [39] and the D-brane quiver theories, constructed in [6]. Let G act on the coordinates (x, y, z) of $\mathbb{C}^3$. Then, the question is: how many algebraically[3] independent polynomials are there of total degree n in (x, y, z). The Molien series is given by

$$M(t;G) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(\mathbb{I}-tg)} = \sum_{i=0}^{\infty} b_i t^i \ , \qquad (3.1)$$

where the determinant is taken over the 3×3 matrix representation of the group elements. Upon series expansion, the coefficients b_i give the number of independent polynomials in degree i. Hence, the f we seek is simply $M(t;G)$.

We can remark one thing immediately. In eq. (3.1) there is only one variable t instead of (x, y, z) in our example $\mathbb{C}^3$. The reason is that for orbifold theories which descend from the $\mathcal{N} = 4$ parent every elementary field has R-charge 2/3. The replacement $x, y, z \to t$ tells us that eq. (3.1) counts the single-trace GIO for given R-charge. Indeed, as a first check, take $G = \mathbb{I}$, the trivial group. We immediately find that

$$M(t;\mathbb{I}) = \frac{1}{\det(\mathbb{I}-t\mathbb{I})} = \frac{1}{(1-t)^3} = 1 + 3\,t + 6\,t^2 + 10\,t^3 + 15\,t^4 + 21\,t^5 + \mathcal{O}(t^6) \ , \qquad (3.2)$$

which agrees with eq. (2.2) for the $\mathbb{C}^3$ theory if one set $t_i = t$. Thus, the Molien series counts invariants of *total* degree in x, y, z whereas eq. (2.2) counts the degree of the three variables individually. In the next subsection, we shall refine the Molien series by straight-forwardly generalising the dummy variable t to a triple $(t_{1,2,3})$.

Emboldened by this check, let us go on to a non-trivial example, the binary dihedral group $\hat{D}_4$ of 8 elements. This is a subgroup of $SU(2) \subset SU(3)$ and is a member of the ADE-series of CY two-fold (K3) singularities (cf. [40]). We can think of this as a

[3]In fact linearly independent, because any polynomial relation would change the total degree. Finding the polynomial relations is a important one and will be subsequently addressed.

$\mathbb{C}^3$ orbifold with one coordinate, say z, held fixed. The gauge theory is the well-known $\mathcal{N}=2$ D-type quiver (q.v. [3, 4, 5, 6]).

This group is generated (we use the standard notation that $\langle x_1,\ldots,x_n\rangle$ is the finite group generated by the list of matrices x_i) as

$$\hat{D}_4 = \langle \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle \ , \tag{3.3}$$

acting on $(x,y)\in\mathbb{C}^2$. We can readily compute the Molien series to be

$$\begin{aligned} M(t,\hat{D}_4) &= \frac{1}{8}(\frac{6}{1+t^2}+\frac{1}{1-2\,t+t^2}+\frac{1}{1+2\,t+t^2}) \\ &= 1+2\,t^4+t^6+3\,t^8+2\,t^{10}+4\,t^{12}+3\,t^{14}+5\,t^{16}+4\,t^{18}+6\,t^{20}+O(t^{22}) \ . \end{aligned} \tag{3.4}$$

This dictates that there are two invariants at degree 4, one at degree 6, etc.

Now, one can actually determine the invariants explicitly, which gives us another check. First, an important theorem due to Nöther states that (cf. e.g. [38]):

THEOREM 3.1. *The polynomial ring of invariants is finitely generated and the degree of the generators is bounded by $|G|$.*

Therefore, though the Molien series is infinite, with increasingly more invariants arising at successive degree with them being linearly independent at each total degree, there will be highly non-trivial algebraic relations amongst the ones at different degree. The power of Theorem 3.1 is that one needs to find invariants at most up to degree equal to the order of the group; all higher degree invariants are polynomials in these basic ones.

Hence, we only need to find a finite number of invariants, which can be determined explicitly due to an averaging technique of O. Reynolds (cf. e.g. [38]). Given any polynomial $F(x)$, one can define the so-called **Reynolds operator**

$$R_G[F(x)] := \frac{1}{|G|}\sum_{g\in G} F(g\circ x) \ . \tag{3.5}$$

Then, the polynomial $R_G[F(x)]$ is invariant by construction. We can then list all monomials of a given degree, apply eq. (3.5) to each and obtain the invariants at the said degree; the number thereof should agree with what eq. (3.1) predicts.

Applying the above discussion to our example of $\hat{D}_4$, we obtain the following invariant polynomials for the first few degrees:

$$\begin{array}{c|c} \text{degree} & \text{invariant polynomials} \\ \hline 4 & x^2y^2, \quad \frac{1}{2}(x^4+y^4) \\ 6 & \frac{1}{2}xy(x^4-y^4) \\ 8 & x^4y^4, \quad \frac{1}{2}x^2y^2(x^4+y^4), \quad \frac{1}{2}(x^8+y^8) \end{array} \tag{3.6}$$

We remark that there are no invariants of lower degree (except trivially the identity) and that the number of independent invariants indeed agree with the series expansion of eq. (3.4). Immediately, one sees some trivial relations such as $x^4y^4 = (x^2y^2)^2$. Using Gröbner basis algorithms [42], one can show that the above ring of 6 invariants can be further reduced to 3. In other words, the ring of invariant polynomials, $\mathbb{C}[x,y]^{\hat{D}_4}$, is generated by 3 so-called **primitive** ones:

$$v = \frac{1}{2}(x^4+y^4), \quad w = x^2y^2, \quad u = \frac{1}{2}xy(x^4-y^4) \ . \tag{3.7}$$

Finding relations among these polynomials is known as the **syzygy problem** and is, again, a classical problem dating to at least Hilbert. The modern solution is, as above, to use Gröbner bases. The reader is referred to [43] for a pedagogical application of syzygies and Gröbner basis to $\mathcal{N}=1$ gauge theories and to [44] in the context of moduli stabilisation. We will return to syzygies later in the paper. For the present example, we find the relation

$$v^2w - w^3 = u^2 \ . \tag{3.8}$$

This is a comforting result. Indeed, invariant theory tells us that

The defining equation of an orbifold is the syzygy of the primitive invariants.

We recognise eq. (3.8) as precisely the defining equation [40] for the affine variety $\mathbb{C}^2/\hat{D}_4$.

3.1.1 ADE-Series

For completeness, let us compute (making extensive use of [42, 45]) the Molien series for the discrete subgroups of $SU(2)$. We find that

$G \subset SU(2)$	$\|G\|$	Generators	Equation	Molien $M(t;G)$
$\hat{A}_{n-1}$	n	$\langle \begin{pmatrix} \omega_n & 0 \\ 0 & \omega_n^{-1} \end{pmatrix} \rangle$	$uv = w^n$	$\frac{(1+t^n)}{(1-t^2)(1-t^n)}$
$\hat{D}_{n+2}$	$4n$	$\langle \begin{pmatrix} \omega_{2n} & 0 \\ 0 & \omega_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle$	$u^2 + v^2 w = w^{n+1}$	$\frac{(1+t^{2n+2})}{(1-t^4)(1-t^{2n})}$
$\hat{E}_6$	24	$\langle S, T \rangle$	$u^2 + v^3 + w^4 = 0$	$\frac{1-t^4+t^8}{1-t^4-t^6+t^{10}}$
$\hat{E}_7$	48	$\langle S, U \rangle$	$u^2 + v^3 + vw^3 = 0$	$\frac{1-t^6+t^{12}}{1-t^6-t^8+t^{14}}$
$\hat{E}_8$	120	$\langle S, T, V \rangle$	$u^2 + v^3 + w^5 = 0$	$\frac{1+t^2-t^6-t^8-t^{10}+t^{14}+t^{16}}{1+t^2-t^6-t^8-t^{10}-t^{12}+t^{16}+t^{18}}$

(3.9)

where we have defined $\omega_n := e^{\frac{2\pi i}{n}}$ and

$$\begin{array}{l} S := \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \quad T := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ U := \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \quad V := \begin{pmatrix} \frac{i}{2} & \frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} \\ -\frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} & -\frac{i}{2} \end{pmatrix} . \end{array} \tag{3.10}$$

We have also used the identity

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{1}{(1-t\omega_n^k)(1-t\omega_n^{-k})} &= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} t^{j+m} n \delta_{j,n\mathbb{Z}} = n \sum_{m=0}^{\infty}\left(\sum_{\beta=0}^{\infty} t^{2m+n\beta} + \sum_{\beta=1}^{\infty} t^{2m-n\beta}\right) \\ &= \frac{n}{1-t^2}\left(\frac{1}{1-t^n} + \frac{1}{t^{-n}-1}\right) . \end{aligned} \tag{3.11}$$

3.1.2 Valentiner: A Non-Abelian $SU(3)$ Example

Having warmed up with the 2-dimensional CY orbifolds, we are ready to study the proper subgroups of $SU(3)$ [6]. The simplest, most well-known, non-trivial, non-Abelian discrete subgroup of $SU(3)$ is perhaps the Valentiner group, otherwise known as $\Delta(3 \cdot 3^2)$

(or, sometimes known as the Heisenberg group for 3 elements, as recently studied in [41]), defined as

$$\Delta(27) := \langle \begin{pmatrix} \omega_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega_3^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle \ . \tag{3.12}$$

Let us investigate this group in some detail; we shall return to this group later in the paper. The Molien series is readily computed to be

$$M(t;\Delta(27)) = \frac{-1+t^3-t^6}{\left(-1+t^3\right)^3} = 1 + 2\,t^3 + 4\,t^6 + 7\,t^9 + 11\,t^{12} + 16\,t^{15} + 22\,t^{18} + \ldots \ . \tag{3.13}$$

To find the defining equation (syzygies), Theorem 3.1 tells us that we need only go up to degree 27 here, a total of 174 invariants, of degrees $0, 3, 6, \ldots, 24, 27$. Using Gröbner techniques [42], we find that there are only 4 nontrivial generators for these 174 polynomials (we have scaled the expressions by an over-all 3):

$$\{m = 3xyz, \quad n = x^3+y^3+z^3, \quad p = x^6+y^6+z^6, \quad q = x^3y^6+x^6z^3+y^3z^6\} \ . \tag{3.14}$$

We then find a single relation in $\mathbb{C}[m,n,p,q]$:

$$8\,m^6 + m^3\,\left(-48\,n^3 + 72\,n\,p + 72\,q\right) + 81\,\left(\left(n^2-p\right)^3 - 4\,n\,\left(n^2-p\right)\,q + 8\,q^2\right) = 0 \ . \tag{3.15}$$

Therefore, $\mathbb{C}^3/\Delta(27)$ is a complete intersection, given by a single (Calabi-Yau) hypersurface in $\mathbb{C}^4$.

3.2 Toric Varieties

Having studied the first class of CY singularities, viz., the orbifolds, in some detail, let us move onto the next, and recently much-revived, class of geometries, the toric singularities. It turns out that here one can also write the Poincaré series f explicitly, now in terms of the combinatorics of the given toric diagram D [37]. Mathematically, this is a nice extension of the Molien series.

We first draw the graph dual[4] of the triangulation of D; this is the (p,q)-web [46], a skeleton of tri-valent vertices indexed by $i \in V$. At each vertex i, the j-th (for

[4] Incidentally, we remark that a convenient way of finding the dual (p,q)-web given a toric diagram is to take, for each pair of toric points, their cross-product, which then gives a vector perpendicular to the plane defined by the two said points.

$j = 1, 2, 3$) of the three coincident edges has charge $\vec{a}_{ij}$ with $\vec{a}$ a three-vector indexed by k, signifying the 3 charges. We remark that toric Calabi-Yau threefolds have three-dimensional toric diagrams whose endpoints are co-planar and this is why D and the dual (p, q)-web are usually drawn on the plane. Here, we need to restore the full coordinates of the 3-dimensional toric diagram; whence, $\vec{a}$ has 3 components. With this notation, the Poincaré series for D is (cf. Eq. (7.24-5) of [37] and also [47] for interesting mathematical perspectives):

$$P(t_1, t_2, t_3; D) = \sum_{i \in V} \prod_{j=1}^{3} \frac{1}{1 - t_1^{a_{ij}^1} t_2^{a_{ij}^2} t_3^{a_{ij}^3}} \ . \tag{3.16}$$

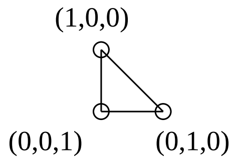
Before we proceed, let us remark on the charges of coordinates t_1, t_2, t_3. For toric varieties, we have three $U(1)$ global symmetries: one is R-charge and the other two, flavor charges. In general, each coordinate t_i is charged under all three $U(1)$. For example, the R-charge of t_2 is given by the inner product of $(0, 1, 0)$ and the **Reeb Vector** $V_R = (b_1, b_2, b_3)$. We recall that in the AdS/CFT correspondence the $U(1)$ R-symmetry is dual to a special Killing vector, the so-called Reeb vector (cf. e.g. [32]), which can be expanded as $V_R = \sum\limits_{i=1}^{3} b_i \frac{\partial}{\partial \phi_i}$, where ϕ_i are the coordinates parametrising the T^3-toric action. It is a very important quantity in toric geometry.

It is possible to make coordinate transformation $(t_1, t_2, t_3) \to (\widetilde{t}_1, \widetilde{t}_2, \widetilde{t}_3)$ such that each coordinate $\widetilde{t}_i$ is charged under one and only one $U(1)$. However, the charges of these new coordinates $\widetilde{t}_i$ in general are not even rational numbers (for example the R-charge of dP_2), so it is not proper to use this new $\widetilde{t}_i$ coordinate to do the Poincaré series expansion, which must have integer powers. Furthermore, as we have seen in the example of $\mathbb{C}^3$, sometimes we want to find the generating function of only one $U(1)$ charge, for example, the R-charge. To do so, we merely make the substitution $(t_1, t_2, t_3) \to (t^a, t^b, t^c)$ for given $a, b, c \in \mathbb{Z}_{\geq 0}$ and the expression will be simplified considerably. In a lot of cases, the interesting $U(1)$ is a linear combination of all three $U(1)$'s as we will see shortly.

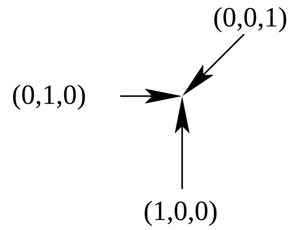
Returning to eq. (3.16), we have some immediate checks. First, we recall that all Abelian orbifolds of $\mathbb{C}^3$ (including $\mathbb{C}^3$ itself) are toric. For example, the toric diagram for $\mathbb{C}^3$ is a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The dual graph, i.e., the (p, q)-web, has a single vertex, with three edges in the directions $(1, 0, 0)$, $(0, 1, 0)$ and

$(0,0,1)$ respectively:

Toric Diagram



(p,q)−web



Hence,

$$\begin{aligned} P(t_1,t_2,t_3;\mathbb{C}^3) &= \frac{1}{1-t_1^1t_2^0t_3^0}\frac{1}{1-t_1^0t_2^1t_3^0}\frac{1}{1-t_1^0t_2^0t_3^1} \\ &= \frac{1}{(1-t_1)(1-t_2)(1-t_3)} = \sum_{i,j,k} t_1^i t_2^j t_3^k \ . \end{aligned} \tag{3.17}$$

This is precisely the result eq. (2.2) obtained from conventional methods in §2.

For a less trivial example, take the conifold $\mathcal{C}$ (cf. [37] as well as an earlier result in [36]). The toric diagram has 4 points, with coordinates

$$A=(0,0,1), \qquad B=(1,0,1), \qquad C=(1,1,1), \qquad D=(0,1,1) \ , \tag{3.18}$$

as shown in the center of Fig. 1. There are two triangulations, giving two (p,q)-webs upon dualising; the two are related by flop transitions. Of course, we need to prove the counting is independent of such choices. Indeed, as the conifold is the building block to all flops in toric varieties, if we show that f is the same for the two (p,q)-webs, this would be true for all toric diagrams and thus we would be at liberty to make any choice of (p,q)-web. First, take the left one, given by the two triangles ABD and BCD. This gives us 2 vertices, with (p,q)-charges of, respectively, $\{(0,1,0),\ (1,0,0),\ (-1,-1,1)\}$ and $\{(0,-1,1),\ (-1,0,1),\ (1,1,-1)\}$. Thus eq. (3.16) gives us

$$P(x,y,z;\ \ \mathcal{C}) = \frac{1}{\left(1-\frac{x\,y}{z}\right)\,\left(1-\frac{z}{x}\right)\,\left(1-\frac{z}{y}\right)} + \frac{1}{(1-x)\,(1-y)\,\left(1-\frac{z}{x\,y}\right)} \ . \tag{3.19}$$

The second trianglization is given by ACD and ABC, giving us

$$\frac{1}{\left(1-\frac{x}{y}\right)\,(1-y)\,\left(1-\frac{z}{x}\right)} + \frac{1}{(1-x)\,\left(1-\frac{y}{x}\right)\,\left(1-\frac{z}{y}\right)} \ . \tag{3.20}$$

It is easy to see that the two expressions eq. (3.19) and eq. (3.20) are the same. Thus indeed the generating function is independent of how we triangulate and how the dual (p,q)-web is obtained [37].

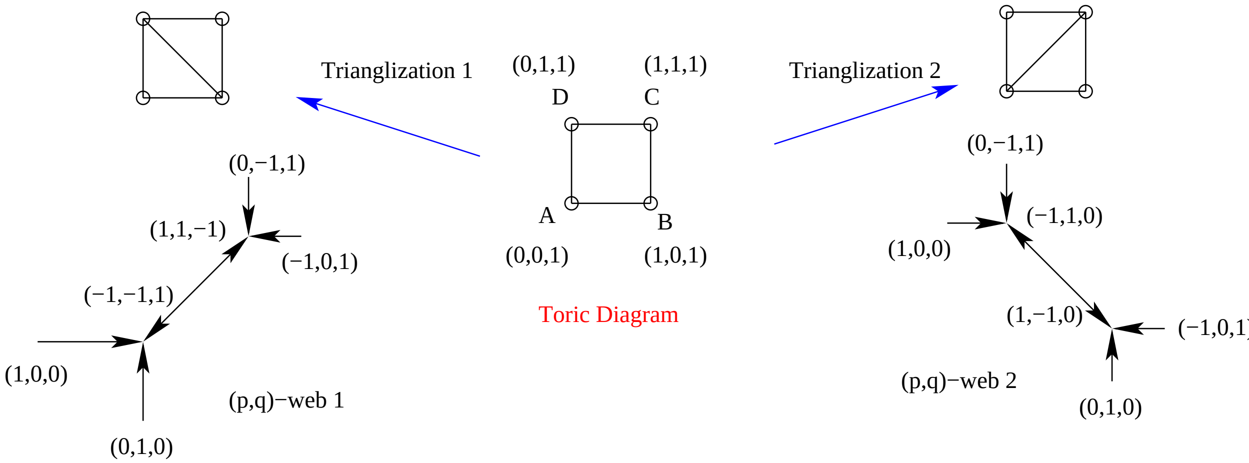


Figure 1: The toric data for the conifold $\mathcal{C}$. There are two triangulations, related to by flops, and thus two (p, q)-webs. We see in the text that they lead to the same counting.

3.2.1 Refinement: $U(1)$-charges and Multi-degrees

We see from eq. (3.16) that for toric varieties the counting is more refined than the Molien series eq. (3.1) as the latter only counts invariants of total-degree. There seems to be a straight-forward generalisation. In order to get the number of single-trace GIO's given the R-charge of each field, the Molien counting seems to be refinable to counting the number of independent polynomials of a given multi-degree $(i_1, \ldots, i_3)$. This is done by generalising the Molien series to:

$$M(t, G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathbb{I} - \mathrm{diag}(t_1, \ldots, t_k) \cdot g)} = \sum_{i_1,\ldots,i_k} b_{i_1,\ldots,i_k} t_1^{i_1} \ldots t_k^{i_k} \ . \tag{3.21}$$

The caveat is that now the coefficients $b_{i_1,\ldots,i_k}$ are no longer guaranteed to be integers for general groups. This corresponds to the fact that the invariants are not monomial in general, but, rather, polynomial. For example, in eq. (3.6), at degree 6, there is a single invariant, built of the sum of two monomials, of multi-degree $(5, 1)$ and $(1, 5)$, each of which is *not* an invariant.

Therefore, this refinement only makes sense in case there is a corresponding conserved charge associated with a $U(1)$ that is part of the isometry of the CY manifold. The isometry of $\mathbb{C}^3$ is $SU(4)$ with rank 3. There is thus a maximum of 3 charges corresponding to the maximal subgroup of the isometry group of the CY manifold. If the manifold is toric then there is a T^3 fibration and therefore a total of three $U(1)$ charges and the index would be a function of 3 variables. If the manifold is not toric then in many cases the isometry group has a rank smaller than 3 and in most cases in fact is absent. Nevertheless there is at least one charge, counting the R charge, that corresponds to the choice of complex structure of the manifold.

To summarize, there are some cases in which the rank of the isometry group is 2 and in most cases the rank is 1. All cases in which the rank is less than 3 are non-toric. An example for a manifold with rank 2 is the set of complete intersection manifolds of the form $x^2 + y^2 + z^2 + w^k = 0$. Is has a clear $SO(3)$ isometry acting on the first 3 coordinates and together with the natural degree of the variables form the isometry group $SU(2) \times U(1)$. For the case $k = 2$ the isometry grows to $SO(4) \times U(1)$ and having rank 3 indeed confirms that the manifold is toric - the familiar conifold. We will study this geometry again in §5.3.2.

An example of a manifold of rank 1 is any $\mathbb{C}^3$-orbifold with a full non-abelian subgroup Γ of $SU(3)$. For Γ in $SU(2)$, we still have $\mathcal{N} = 2$ SUSY, so the global isometry is $SU(2) \times U(1)$, of rank 2. Indeed though there is no refinement for 3 charges, there should still be a refinement of two charges since the rank is 2. The first charge will denote the Cartan charge of the $SU(2)$. We could, for example, define a degree which counts how many x's and y's together, and another degree which counts how many z's. This is the reason that in trying to implement the refinement on $\hat{D}_4$ we found factors of 1/2. There is simply no corresponding conserved charge which corresponds to this generalization.

3.2.2 The $Y^{p,q}$ Family

An infinite family of toric CY 3-folds which has recently attracted much attention, because of the discovery of explicit CY metric thereon, is the $Y^{p,q}$'s (cf. e.g.,[48, 49, 50]). Let us now do the counting for these. The toric data is given by $O = (0, 0, 1)$, $A = (1, 0, 1)$, $B = (0, p, 1)$ and $C = (-1, p - q, 1)$. We have drawn it at the left hand side of Fig. 2. As indicated, we take the trianglization as connecting the point $T_a = (0, a, 1)$ to A and C with $a = 1, ..., p$ (so $T_p = B$). Now we have $2p$ triangles given by $T_a A T_{a+1}$ and $T_a C T_{a+1}$, $a = 0, ..., p - 1$. For triangle $T_a A T_{a+1}$ we have the following charges and corresponding term

$$\{(1, 0, 0), \quad (a, 1, -a), \quad (-a - 1, -1, a + 1)\} \Rightarrow \frac{1}{(1 - x)\left(1 - \frac{x^a y}{z^a}\right)\left(1 - \frac{x^{-1-a} z^{1+a}}{y}\right)} . \tag{3.22}$$

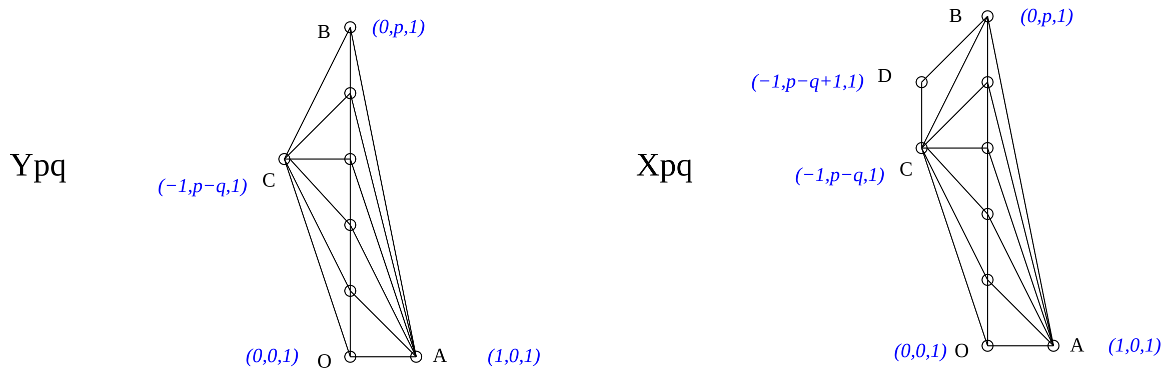


Figure 2: The toric diagrams for the spaces $Y^{p,q}$ and $X^{p,q}$.

For triangle T_aCT_{a+1} we have

$$\{(-1,0,0),\quad (-a+(p-q),1,-a),\quad ((q-p)+a+1,-1,a+1)\} \Rightarrow \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{x^{-a+p-q}\,y}{z^a}\right)\left(1-\frac{x^{1+a-p+q}\,z^{1+a}}{y}\right)}\,. \tag{3.23}$$

Putting these together we have

$$f(x,y,q;\,Y^{p,q}) = \sum_{a=0}^{p-1} \frac{1}{(1-x)\,\left(1-\frac{x^a\,y}{z^a}\right)\left(1-\frac{x^{-1-a}\,z^{1+a}}{y}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{x^{-a+p-q}\,y}{z^a}\right)\left(1-\frac{x^{1+a-p+q}\,z^{1+a}}{y}\right)}\,. \tag{3.24}$$

Knowing $Y^{p,q}$, it is easy to go on to $X^{p,q}$. It differs therefrom by the addition of one point $(-1,p-q+1,1)$ in the toric diagram. So we use the above trianglization of $Y^{p,q}$, plus one more triangle given by $(0,p,1)$, $(-1,p-q+1,1)$ and $(-1,p-q,1)$. This one gives the following vector and hence a new term to eq. (3.24):

$$\{(1,0,1),(q-1,-1,p),(-q,1,-p)\} \quad \Rightarrow \quad \frac{1}{(1-x\,z)\,\left(1-\frac{y}{x^q\,z^p}\right)\left(1-\frac{x^{-1+q}\,z^p}{y}\right)}\,.$$

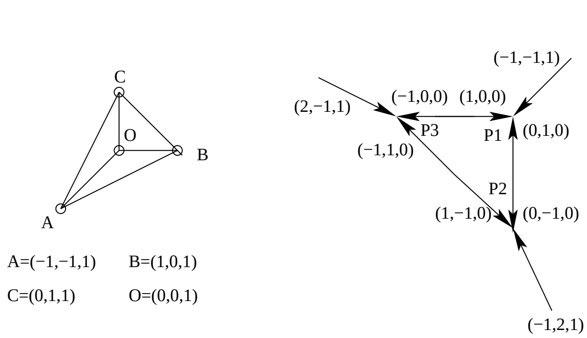
3.3 The Del Pezzo Family

The last category of CY singularities widely studied in D-brane gauge theories is the cone over the 9 del Pezzo surfaces. These surfaces are $\mathbb{P}^2$ blown up at n generic points; the cone is CY if $n=0,\ldots,8$. There is a close cousin to this family, viz, the zeroth Hirzebruch surface F_0, which is simply $\mathbb{P}^1\times\mathbb{P}^1$ and the cone over which is also CY. It is well-known that for $dP_{n=0,1,2,3}$ and for F_0, the space is actually toric (q.v. [11]).

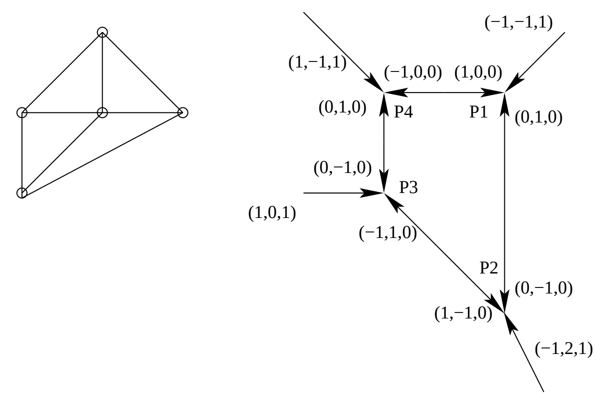
Therefore, we can use eq. (3.16) to obtain the following:

$$
\begin{aligned}
P(z,x,y;dP_0) &= \frac{1}{(1-x)\,(1-y)\left(1-\frac{z}{x\,y}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{y}{x}\right)\left(1-\frac{x^2\,z}{y}\right)} + \frac{1}{\left(1-\frac{1}{y}\right)\left(1-\frac{x}{y}\right)\left(1-\frac{y^2\,z}{x}\right)}\\
P(z,x,y;F_0) &= \frac{1}{(1-x)\,(1-y)\left(1-\frac{z}{x\,y}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)(1-y)\left(1-\frac{x\,z}{y}\right)} + \frac{1}{(1-x)\left(1-\frac{1}{y}\right)\left(1-\frac{y\,z}{x}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)(1-x\,y\,z)}\\
P(z,x,y;dP_1) &= \frac{1}{(1-x)\left(1-\frac{y}{x}\right)\left(1-\frac{z}{y}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)(1-y)\left(1-\frac{x\,z}{y}\right)} + \frac{1}{(1-x)\left(1-\frac{x}{y}\right)\left(1-\frac{y\,z}{x^2}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)(1-x\,y\,z)}\\
P(z,x,y;dP_2) &= \frac{1}{\left(1-\frac{x}{y}\right)(1-y)\left(1-\frac{z}{x}\right)} + \frac{1}{(1-x)\left(1-\frac{y}{x}\right)\left(1-\frac{z}{y}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)(1-y)\left(1-\frac{x\,z}{y}\right)} +\\
&\quad + \frac{1}{(1-x)\left(1-\frac{1}{y}\right)\left(1-\frac{y\,z}{x}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)(1-x\,y\,z)}\\
P(z,x,y;dP_3) &= \frac{1}{\left(1-\frac{1}{y}\right)(1-x\,y)\left(1-\frac{z}{x}\right)} + \frac{1}{\left(1-\frac{1}{x\,y}\right)(1-y)\,(1-x\,z)} + \frac{1}{\left(1-\frac{1}{x}\right)(1-x\,y)\left(1-\frac{z}{y}\right)} +\\
&\quad + \frac{1}{(1-x)\,(1-y)\left(1-\frac{z}{x\,y}\right)} + \frac{1}{(1-x)\left(1-\frac{1}{x\,y}\right)(1-y\,z)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)(1-x\,y\,z)}\ .
\end{aligned}
\tag{3.25}
$$

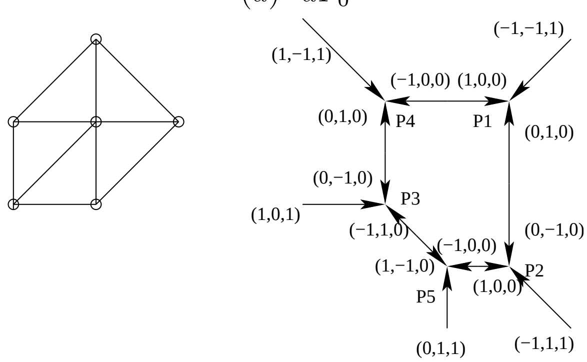
We include toric diagrams and the dual (p,q)-webs here for reference:



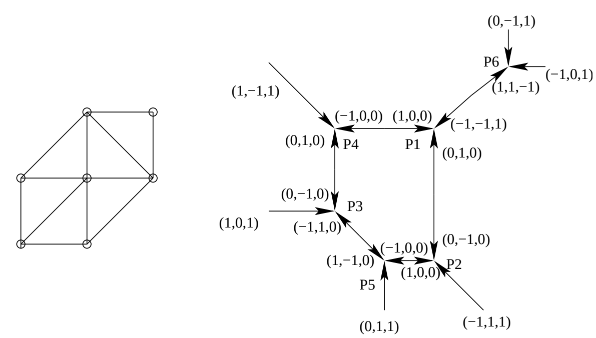
(a) dP_0



(b) dP_1

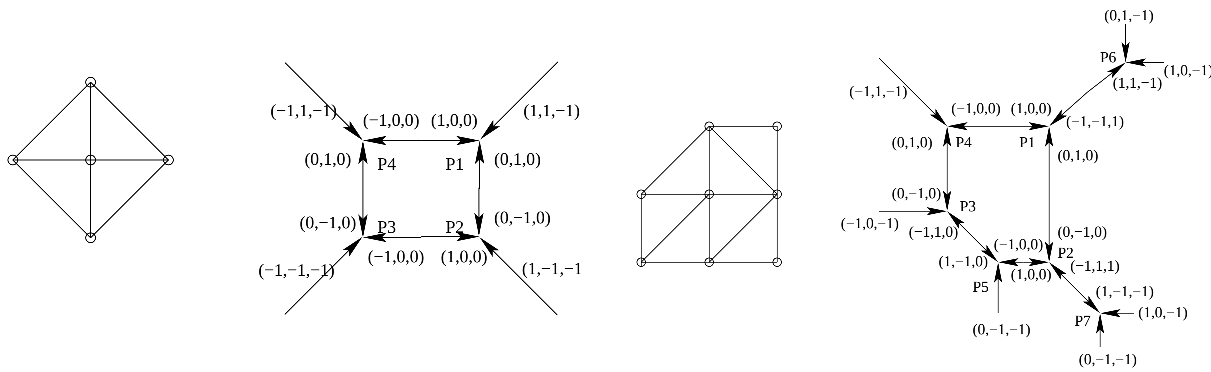


(c) dP_2



(d) dP_3

For completeness, we also include the data for F_0 as well as $PdP4$, the so-called pseudo dP_4 surface, first introduced in [51], which is obtained from blowing up a non-generic point of dP_3 so as to keep it a toric variety:



$(a)\ \ F_0$ $\qquad\qquad$ $(b)\ \ PdP_4$

3.3.1 A General Formula for dP_n

We can see that taking the limit $x = y \to 1$ to relax the refinement in eq. (3.25) the expressions become very simple. In other words, we neglect the two $U(1)$ charges carried by x, y and keep only the $U(1)$ charge carried by $t = z$ (note that this is not the R-charge but a linear combination of the three $U(1)$ charges). The result counts the single-trace GIO's of a given total degree:

$$\begin{array}{ll} f(t;dP_0) = \frac{1+7t+t^2}{(1-t)^3}, & f(t;dP_1) = f(t;F_0) = \frac{1+6t+t^2}{(1-t)^3}, \\ f(t;dP_2) = \frac{1+5t+t^2}{(1-t)^3}, & \qquad f(t;dP_3) = \frac{1+4t+t^2}{(1-t)^3}\ . \end{array} \tag{3.26}$$

We shall see in the next section what it means to set x, y to 1 and how all this relates to projecting 3-dimensional toric diagrams to 2-dimensions and to dimers. For the mean time, observing the pattern eq. (3.26) for the above 4 members of the del Pezzo family, we propose the following general expression for the generating function:

$$f(t)^{(n)} := f_\infty(t;dP_n) = \frac{1+(7-n)t+t^2}{(1-t)^3}, \qquad n = 0,\ldots,8\ . \tag{3.27}$$

We remark that the result for F_0 is the same as dP_1. This is not surprising because they both, when having 1 more generic point blown-up, become dP_2. Also, dP_0 is a Calabi-Yau over $\mathbb{P}^2$, it is in fact simply the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ which we will encounter again in §5.1.2. Furthermore, setting $n = 4$ gives agreement with the recent $(P)dP_4$ [51] result of Eq. 5.29 of [27]. Indeed, we shall revisit the del Pezzo family, and give full credence to eq. (3.27) in §5.3.1.

4 Dimers, Toric Diagrams and Projections

By now we have seen the Poincaré series f_∞ in full action in counting single-trace GIO's. Before proceeding to finding the generating function g_∞ for the multi-trace case, let us take a brief but important interlude in how the counting in f is pictorially realised for toric varieties. In due course, we shall see how the invariants emerge in slices of the 3-dimensional toric cones and how such projections relate to dimers and 2-dimensional tilings. Indeed, it is these observations which initiated our original interest in this problem of counting GIO's.

4.1 Example: Dimers and Lattices for $\mathbb{C}^3$

We begin by discussing the simplest toric CY 3-fold, $\mathbb{C}^3$, which was first mentioned in §2 and then in §3.2. Let us see how to represent the chiral ring in the dimer diagram of $\mathbb{C}^3$. The dimer for $\mathbb{C}^3$ is well-known [16] and is drawn in Fig. 3. There is only one gauge group and it is represented by a hexagon. We recall the fundamental fact that in a dimer, the polygonal faces correspond to gauge groups, edges, (perpendicular) to fields and nodes, to superpotential terms. Thus, a BPS GIO in the chiral ring [5] can be thought of as a path from the origin to a polygon, and we shall show below that it is in fact path-independent. Since we consider only chiral operators, we represent the (holomorphic) operator by oriented lines crossing edges, such that when a line crosses an edge, the black vertex is to its left (recall that the coloring convention in a dimer has orientation built in). There are three holomorphic fields denoted by X, Y, Z. As mentioned before, the F-term relations here make these three operators mutually commutative.

Now, we discuss the chiral GIO's in detail. We shall do so according to the number of **levels**. Here, we define level to mean the number of X, Y, Z fields inside the chiral operators. This was what we meant by *degree* in the aforementioned generating function. For clarity, we have enclosed each level with a dotted red circle in the diagram. At level 1, there are only 3, given by $\mathrm{Tr}(X)$, $\mathrm{Tr}(Y)$ and $\mathrm{Tr}(Z)$. This has been shown in level one of Fig. 3. In the figure we have given also the 3 anti-chiral operators $\mathrm{Tr}(\bar{X})$, $\mathrm{Tr}(\bar{Y})$ and $\mathrm{Tr}(\bar{Z})$ for reference. Henceforth, we shall use *blue* to denote the GIO's in which we are interested, viz., the chiral single-traces ones. The 3 here, of course, correspond to the

[5] Since we are studying BPS mesons, we may emphasize the relationship between the concept of the "extremal BPS meson" (introduced in [28]) and that of the "zig-zag path" in [21, 22, 24].

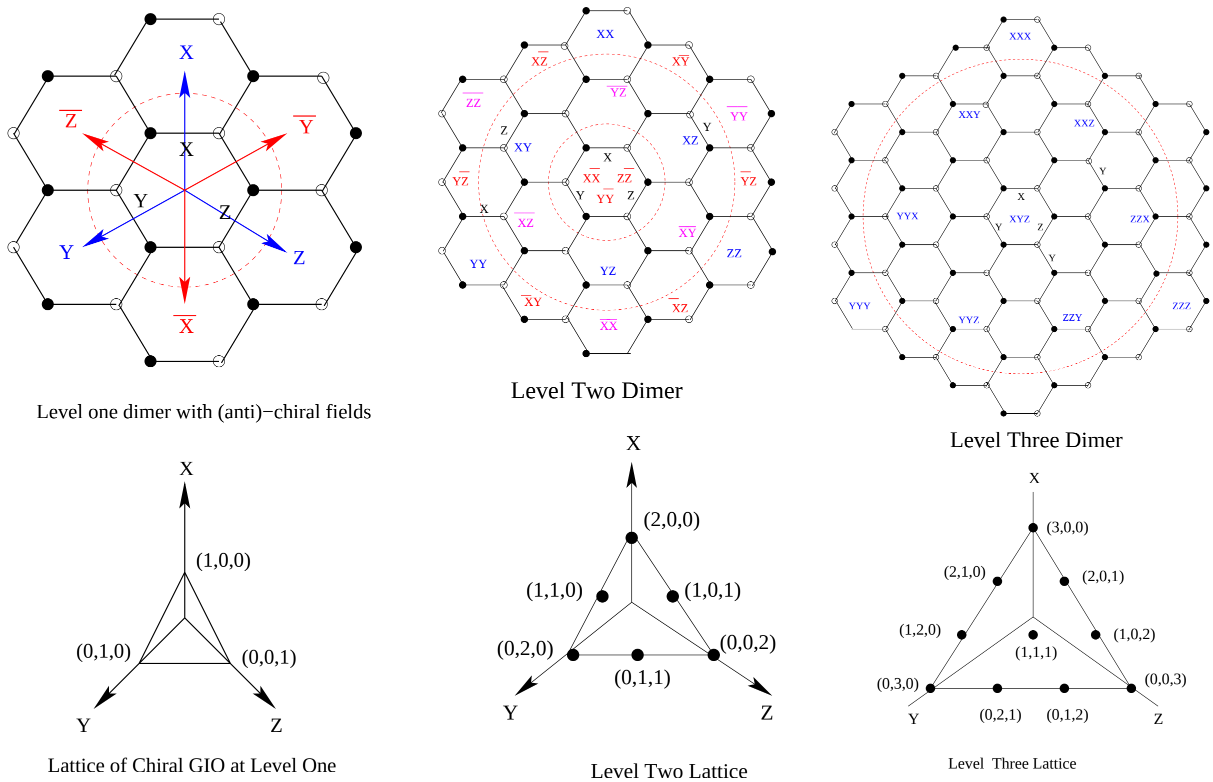


Figure 3: The Dimer configurations and the lattice structure of GIO's for $\mathbb{C}^3$, exhibited at the first 3 levels. We have drawn some mixed chiral-antichiral GIO's as well for illustration, but the ones of our concern, viz., the chiral ones, are drawn in blue.

$3t$ term in eq. (3.2).

Next, let us move to level 2 of Fig. 3. This time, we can cross two edges, as shown by the second red circle. A few remarks are at hand. First, for the hexagon denoted by the blue XY (which is inside the chiral ring), we have two paths to reach from the center. One is from the center to hexagon $\overline{YZ}$ then to XY. Another one is from the center to $\overline{XZ}$ then to XY. The key point is that these two paths give the same element $\mathrm{Tr}(XY)$ in the chiral ring. So, our first conclusion is that for the chiral ring, a GIO depends only on the starting and ending point of the path in the dimer model and does not depend on the path itself. This in fact is generic for every dimer model and not just for the simple hexagonal model discussed here. See [21, 24, 28] for a proof of this point.

Furthermore, we have in fact drawn not only the chiral ring, but also the anti-chiral ring and the mixed chiral-antichiral operators[6] at level 2. For the mixed operators, it

[6]We remark that the mixed operators are protected only in $\mathcal{N} = 4$ because of enhanced SUSY, in generic $\mathcal{N} = 1$ theories the "protected rings" are only the chiral or antichiral ones. The other 1/2-BPS

is easy to see that now the path in the dimer does matter. We start from the center, go up and then go down and get $\bar{X}X$. Similarly we can go southwest to get $\bar{Y}Y$ and southeast to get $\bar{Z}Z$. This means that we should put all three $\bar{X}X$, $\bar{Y}Y$ and $\bar{Z}Z$ in the center hexagon. In another word, the one-to-one correspondence we found for chiral or anti-chiral ring is lost. Because of this complexity we will not discuss mixed operators further in this paper (the anti-chiral ring is isomorphic to the chiral ring and need not be addressed separately).

Focusing on only the chiral ring we can see that there are 6 (chiral) GIO's at level 2, corresponding to the term $6t^2$ in eq. (3.2). Also, The result of level 3 is given at the right of Fig. 3. Here, we give only the chiral ring operators at proper hexagons in this figure. There is a total of 10 as shown, corresponding to the $10t^3$ in eq. (3.2).

4.1.1 Lattice Structure and Planar Slices

We thus conclude that:

> A (chiral) GIO's at level n corresponds to a polygon in the dimer, which is a chiral-distance n away from the center.

In the above, a chiral-distance is measured by segments of only chiral operators, i.e., black vertices to the left. We have drawn these chiral GIO's in blue in Fig. 3. With this one-to-one correspondence between chiral GIO's at a given level and hexagons in the dimer diagram, we can see that in fact we have a lattice structure in $\mathbb{R}^3$. Each integer lattice point (a, b, c) with $a, b, c \geq 0$ corresponds to a chiral operator. The level of this operator is given by $(a + b + c)$. In other words, level n is given by the plane perpendicular to vector $(1, 1, 1)$ and has distance n from the origin. It is interesting to notice that the vector $(1, 1, 1)$ is the Reeb vector of $\mathbb{C}^3$. Thus the degree we are counting is exactly the R-charge. We have drawn these plane slices for each level in our figure as well in Fig. 3.

This lattice is something with which we are familiar! It is nothing other than the dual toric cone for $\mathbb{C}^3$. Indeed, the definition of a toric variety is that it is the affine spectrum of the ring of monomials obtained from raising the coordinates to the powers of the lattice generators, i.e., the invariant monomials. This is what we are doing above. Level 1 gives the monomials which are obtained from the lattice generators of the cone; level

protected operators like the currents do not form a ring.

2 gives the monomials obtained from the toric cone intersected with the (non-primitive) lattice points one further step away, etc.

4.2 Example: Dimers and Lattices for the Conifold

Next, let us discuss the conifold. Here, we will see explicitly how we must count the GIO's up to relations from F-terms, as was mentioned in the introduction. The toric diagram was given in Fig. 1 in §3.2. The dimer model is the brane diamond [52] drawn on T^2 [16] and is given in Fig. 4. There are two gauge groups so there will be two types of polygons which are labeled 1 and 2. It is easy to see that we can locate these 2 gauge groups at lattice points. More explicitly, if we draw the lifting of T^2 in $\mathbb{R}^2$, we can identify an integer lattice point (a, b) to gauge group 1 if a, b are integers or gauge group 2 if a, b are half-integers. Now, since we are considering the single-trace mesonic GIO's, we can neglect gauge group 2 and consider the holomorphic paths connecting different lattice points of gauge group 1, i.e., integer lattice points in 2-dimensions.

In this gauge theory, there are four bi-fundamental fields A_1, A_2 and B_1, B_2. We define the following operators which are in the adjoint representation of gauge group 1:

$$M_{0,1} = A_1 B_1, \qquad M_{1,0} = A_1 B_2, \qquad M_{-1,0} = A_2 B_1, \qquad M_{0,-1} = A_2 B_2 \ . \tag{4.1}$$

It is easy to check that the F-term relations tell us that all four M_{ij} commute and obey one non-trivial relation:

$$M_{0,1} M_{0,-1} = M_{1,0} M_{-1,0} \ . \tag{4.2}$$

We can map the above quantities into the dimer model. As we have shown above, the dimer model can be mapped to a 2-dimensional integer lattice. The operator $M_{0,1}$ can be mapped to vector $(0, 1)$ so we can use it to connect points $(0, 0)$ and $(0, 1)$. Similarly, $M_{1,0}, M_{-1,0}, M_{0,-1}$ map to vectors $(1, 0), (-1, 0), (0, -1)$, respectively. Using this mapping, a single-trace GIO is mapped to a path connecting point $(0, 0)$ to (a, b) using the above four vectors. The non-trivial relation eq. (4.2) is nothing, but the statement that after following consecutively vectors $(0, 1)$ and $(0, -1)$ (or $(1, 0)$ and $(-1, 0)$) we come back to the starting point.

Using this picture we can see the lattice structure of holomorphic GIO's. For level 0, it is the origin $(0, 0)$ and corresponds to the identity operator. For level one, we can use only one $M_{i,j}$ to connect $(0, 0)$ to nearby lattice points. Thus we have four of them $\mathrm{Tr}(M_{1,0})$, $\mathrm{Tr}(M_{-1,0})$, $\mathrm{Tr}(M_{0,1})$, and $\mathrm{Tr}(M_{0,-1})$. For level two, we need to use two

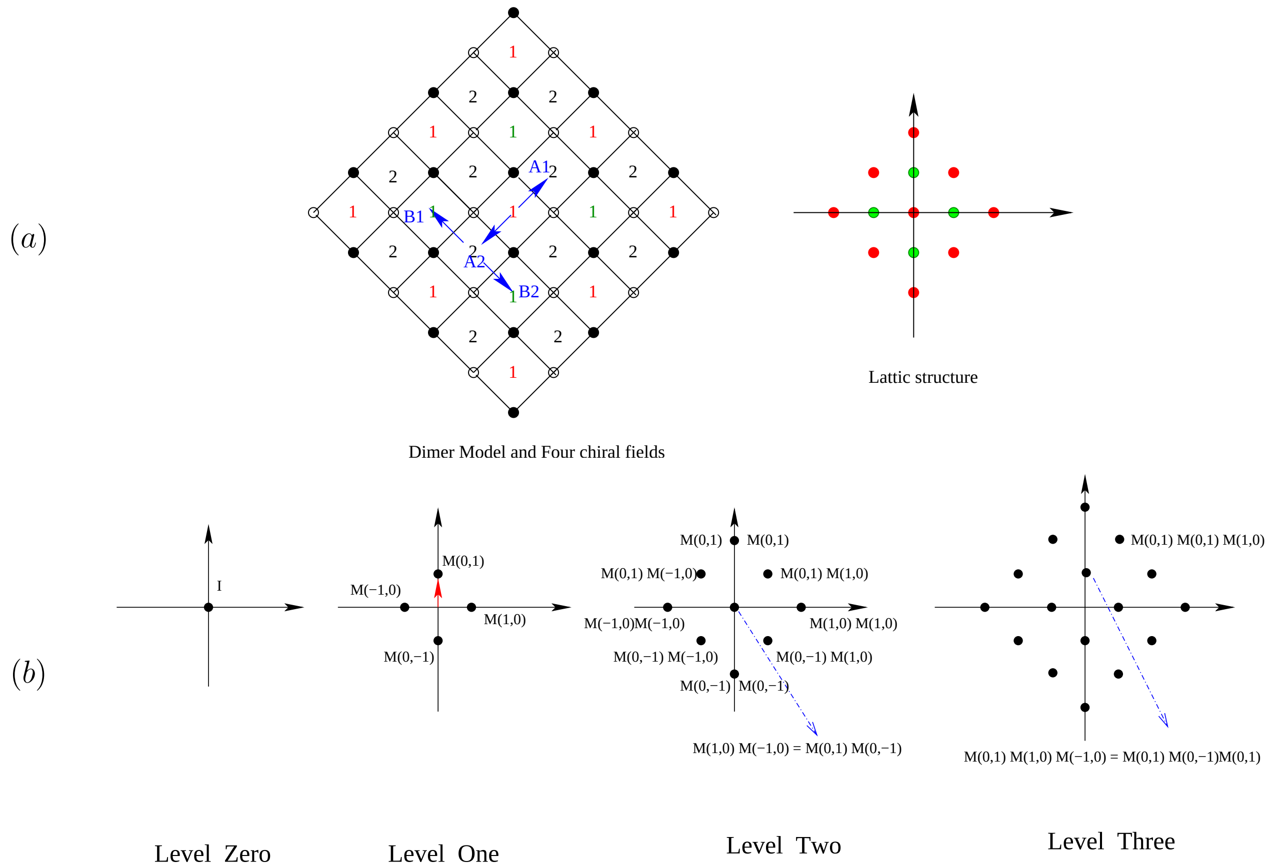


Figure 4: (a) The dimer configuration for the conifold and the dual lattice structure of GIO's; (b) In more detail, the actual operators (with built-in relations) corresponding to the lattice points at the first 3 levels.

$M_{i,j}$ operators. It is easy to get to lattice points $(\pm 2, 0), (\pm 1, \pm 1)$ as well as $(0, 0)$. For $(0, 0)$ we have two ways $\mathrm{Tr}(M(0,1)M(0,-1))$ or $\mathrm{Tr}(M(1,0)M(-1,0))$. But by relation eq. (4.2) they are the same so we should count only once. Similarly we can draw the level three lattice diagram as shown in Fig. 4.

4.2.1 Planar Slices and Lattices

Now let us find the 3-dimensional box which projects to the above 2-dimensional picture. The vectors $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ of the toric diagram of $\mathcal{C}$ generates an integral cone; we can find the generators of the dual cone to be $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, -1, 1)$ and $v_4 = (-1, 0, 1)$. By definition, a lattice point in the dual cone is given by positive integer linear combinations of these four vectors. It is special in our case that these four generators v_i have their endpoints co-planar[7]. It is easy to find the vector u orthogonal to the plane generated by v_i as $u = (1, 1, 2)$. In fact, in this case, the vector u is precisely the Reeb vector, so the level we are counting is also the R-charge[8].

Now, we can see how this 3-dimensional lattice generated by v_i projects to the 2-dimensional lattice. For level one, it is given by four v_i, since all of them have $v_i \cdot (1, 1, 2) = 1$. For the level two, we need to find vectors (x, y, z) such that (1) $(x, y, z) = \sum_{i=1}^4 a_i v_i$ with $a_i \geq 0$ and integer; (2) $(x, y, z) \cdot (1, 1, 2) = 2$ which gives $x + y + 2z = 2$. From these conditions, we find the following 9 points:

$$\{(2,0,0),(0,2,0),(0,-2,2),(-2,0,2),(1,1,0),(1,-1,1),(0,0,1),(-1,-1,2),(-1,-1,2)\} \tag{4.3}$$

which is exactly what we find in the dimer model. In general level n should have $(n+1)^2$ points.

Now let us check this using our Poincaré series, which from eq. (3.19) is

$$P(x, y, z;\ \mathcal{C}) = -\frac{x\, y\ (-1 + z)}{(-1 + x)\ (-1 + y)\ (x - z)\ (y - z)}\ . \tag{4.4}$$

To count the level, notice that the Reeb vector is $u = (1, 1, 2)$, which means that the R-charges of x, y, z are $1, 1, 2$ respectively. In other words, we should replace $x \to q$, $y \to q$, $z \to q^2$, yielding

$$P(q;\ \mathcal{C}) = \frac{(1 + q)}{(1 - q)^3} = \sum_{n=0}^{\infty} (n + 1)^2 q^n\ ; \tag{4.5}$$

whereby giving us the required $(n + 1)^2$ counting!

In fact we can do better than that. Let us do the following replacement $x \to xq$, $y \to yq$ and $z \to q^2$. The expression is changed to

$$\frac{xy(1 - q^2)}{(1 - qx)(1 - qy)(q - x)(q - y)} = 1 + q(x + y + \frac{1}{x} + \frac{1}{y}) + \dots\ . \tag{4.6}$$

Comparing this with toric data we can see that x, y represent the Cartan weight of $SU(2)_L \times SU(2)_R$ global symmetry for the conifold. More explicitly, for the two $U(1) \times U(1)$, x, y carry the charge of $U(1)_x = U(1)_L + U(1)_R$ and $U(1)_y = U(1)_L - U(1)_R$. To see this let us consider, for example, $M_{0,1} = A_1 B_1$. Because the $(U(1)_L, U(1)_R)$ charge of

[7]Note that though the toric diagram always has its vectors in a plane, as guaranteed by the CY condition, the dual cone is not so guaranteed.

[8]We remark that our Reeb vector differs in convention from that of [37]. Our (a, b, c) is their (c, a, b).

A_1 and B_1 is $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, we get immediately the $(U(1)_x, U(1)_y)$ charge $(1, 0)$, i.e., the term x. Similarly, the terms $y, \frac{1}{x}, \frac{1}{y}$ correspond to the operators $M_{0,1}, M_{-1,0}, M_{0,-1}$, respectively.

5 Counting Gauge Invariants: Plethystics, Multi-Trace and Syzygies

We have now seen the generating function f which counts single-trace GIO's of a given choice of global charges for 3 large families of CY threefold singularities. What about the multi-trace GIO's? These are products of combinations of single-traces. We have called the generating function for counting these, g. We shall now see how g can be obtained from f using some nice combinatorics. We shall then see how the function which relates f and g has some remarkable geometrical properties as well.

5.1 The Plethystic Exponential: From Single to Multi-Trace

Recall that in the above f should really be f_∞ bacause we have taken the large N limit. Similarly, the quantity g we desire is really g_∞. Now, we showed in §2 that for $\mathbb{C}^3$, the relation between f and g is that of the **plethystic exponential**, PE (q.v. [53, 54]). This in fact holds in general:

$$g(t) = PE[f(t)] := \exp\left(\sum_{k=1}^{\infty} \frac{f(t^k) - f(0)}{k}\right) . \tag{5.1}$$

Indeed, recalling eq. (2.6), we summarise the following relations, with the subscripts restored:

$$g_1 = f_\infty, \qquad f_\infty(t) = PE[f_1(t)], \qquad g_\infty(t) = PE[g_1(t)] = PE[PE[f_1(t)]] . \tag{5.2}$$

We remark that, even for a list of variables $t_{i=1,\ldots,n}$, which are used in the refinement of counting discussed above, the expressions in eq. (5.1) and eq. (5.2) still hold, with obvious replacement. Namely,

$$g(t_1, \ldots, t_n) = PE[f(t_1, \ldots, t_n)] := \exp\left(\sum_{k=1}^{\infty} \frac{f(t_1^k, \ldots, t_n^k) - f(0, \ldots, 0)}{k}\right) . \tag{5.3}$$

We can derive the statement eq. (5.1) explicitly by series-expansion. Let

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{5.4}$$

be the Taylor expansion of the Poincaré series $f_\infty = f(t)$. Thus, a_n is the number of independent invariants at (total) degree n. Then, eq. (5.1) gives us

$$\begin{aligned} PE[f(t)] &= \exp\left(\sum_{n=0}^{\infty} a_n \sum_{k=1}^{\infty} \frac{t^{nk}}{k} - a_0 \sum_{k=1}^{\infty} \frac{1}{k}\right) \\ &= \exp\left(-\sum_{n=0}^{\infty} a_n \log(1-t^n) - a_0 \sum_{k=1}^{\infty} \frac{1}{k}\right) \ . \end{aligned}$$

We see therefore that the $f(0)$ term precisely regularises the sum and we obtain

$$PE[f(t)] = \exp\left(-\sum_{n=1}^{\infty} a_n \log(1-t^n)\right) = \frac{1}{\prod\limits_{n=1}^{\infty} (1-t^n)^{a_n}} \ . \tag{5.5}$$

This expression is now in the standard Euler product form. Upon expansion of $PE[f(t)]$, we would see that the coefficient for t^m is the number of ways of partitioning m, each weighted by a_n. This is precisely our required counting, i.e., the number of multi-trace GIO's at degree m. Hence, $g(t) = PE[f(t)]$.

We have thus solved problems (1) and (2) posed in the introduction and have the generating functions f and g for large N. In fact, as before, we can refine our counting. In the above, we had a single variable t, a dummy variable associated with the total degree. Where permitted, as discussed in §3.2.1, we can have a set of variables t_i, one for each $U(1)$-charge, and an associated multi-degree for these tuples of charges. In addition, we can introduce one more variable ν, to be inserted into the summand. One could easily see that upon expansion, the power of ν will actually count how many single-trace operators are present in each of the terms. In other words, for $f_\infty(t_1, \ldots, t_m) = \sum\limits_{p_1,\ldots,p_m=0}^{\infty} a_{p_1,\ldots,p_m} t_1^{p_1} \ldots t_m^{p_m}$, we have

$$\begin{aligned} \tilde{g}_\infty(t_i, \nu) &= PE[f_\infty] = \exp\left(\sum_{k=1}^{\infty} \frac{f_\infty(t_1^k, \ldots, t_m^k)\nu^k}{k}\right) \\ &= \left(\prod_{p_1,\ldots,p_m} (1 - \nu t_1^{p_1} \ldots t_m^{p_m})^{a_{p_1,\ldots,p_m}}\right)^{-1} \ ; \end{aligned} \tag{5.6}$$

note that due to the insertion of ν, there is no longer a need to regulate the sum by the subtraction of $f(0, \ldots, 0)$.

5.1.1 The Plethystic Logarithm

The inverse function of PE is also a fascinating one. It is called the plethystic logarithm [53]; one can in fact write it analytically:

$$f(t) = PE^{-1}(g(t)) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g(t^k)) \ , \tag{5.7}$$

where $\mu(k)$ is the Möbius function

$$\mu(k) = \begin{cases} 0 & k \text{ has one or more repeated prime factors} \\ 1 & k = 1 \\ (-1)^n & k \text{ is a product of } n \text{ distinct primes} \end{cases} \ . \tag{5.8}$$

As $g_\infty = PE[g_1]$, so too does one have the relation $f_\infty = PE[f_1]$. Since our basic generating function is the Poincaré series $f = f_\infty$, for which we have had explicit results in §3, it is more convenient to write

$$f_1 = PE^{-1}(f_\infty) \ . \tag{5.9}$$

One may ask what this function f_1, which we briefly encountered in eq. (2.5), signifies. It has a remarkable geometrical property!

> The plethystic logarithm of the Poincaré series, is a generating series for the relations and syzygies of the variety!

We exemplify this statement with our familiar example of the Valentiner group from §3.1.2 in the next subsection.

5.1.2 Plethystic Logarithm and Syzygies

Using eq. (5.7), and recalling the Poincaré (Molien) series f for $\Delta(27)$ from eq. (3.13), we see that

$$f_1 = PE^{-1}\left(\frac{-1+t^3-t^6}{\left(-1+t^3\right)^3}\right) = 2t^3 + t^6 + t^9 - t^{18} \ . \tag{5.10}$$

The RHS terminates and is a polynomial! It is to be interpreted thus: there are 2 degree 3 invariants, 1 degree 6 and 1 degree 9 invariant, these 4 invariants obey a single relation of total degree 18. Upon inspecting eq. (3.14) and eq. (3.15), we see that this is indeed the definition of $\mathbb{C}^3/\Delta(27)$ as a variety!

Now, $\mathbb{C}^3/\Delta(27)$, as a hypersurface in $\mathbb{C}^4$, is a complete intersection affine variety (i.e., the number of equations is equal to the codimension of the variety in the embedding space). How does the above work for non-complete intersections? We have an example readily available: the famous $\mathbb{C}^3/\mathbb{Z}_3 = \mathcal{O}_{\mathbb{P}^2}(-3)$ orbifold. In fact, being an Abelian orbifold, this is also toric and furthermore, it is also dP_0, being a cone over $\mathbb{P}^2$. So we have 3 ways to compute its Molien series from §3. Let us use the Molien series. The action is $(x, y, z) \to \omega_3(x, y, z)$ and we immediately get

$$f_\infty(t) = M(t; \mathbb{Z}_3) = \frac{1 + 7\,t^3 + t^6}{(1 - t^3)^3} \ , \tag{5.11}$$

whereby

$$f_1(t) = PE^{-1}[f_\infty(t)] = 10\,t^3 - 27\,t^6 + 105\,t^9 - 540\,t^{12} + 3024\,t^{15} - 17325\,t^{18} + \mathcal{O}(t^{21}) \ . \tag{5.12}$$

This is again in accordance with known facts! The equation for this orbifold is 27 quadrics in $\mathbb{C}^{10}$, i.e., 10 degree 3 invariants satisfying 27 relations of degree $2 \times 3 = 6$ (q.v. [38, 43]). We can determine these as follows. The 10 invariants are

$$y_{1,\ldots,10} = \left\{x^3,\ x^2y,\ xy^2,\ y^3,\ x^2z,\ xyz,\ y^2z,\ xz^2,\ yz^2,\ z^3\right\} \ , \tag{5.13}$$

obeying the 27 quadrics

$$\begin{aligned}
&\{{y_2}^2 - y_1y_3,\ y_2y_3 - y_1y_4,\ {y_3}^2 - y_2y_4,\ y_2y_5 - y_1y_6,\ y_3y_5 - y_1y_7,\ y_2y_6 - y_1y_7,\\
&y_4y_5 - y_2y_7,\ y_3y_6 - y_2y_7,\ y_4y_6 - y_3y_7,\ {y_5}^2 - y_1y_8,\ y_5y_6 - y_1y_9,\ y_2y_8 - y_1y_9,\\
&{y_6}^2 - y_2y_9,\ y_5y_7 - y_2y_9,\ y_3y_8 - y_2y_9,\ y_6y_7 - y_3y_9,\ y_4y_8 - y_3y_9,\ {y_7}^2 - y_4y_9,\\
&y_5y_8 - y_1y_{10},\ y_6y_8 - y_2y_{10},\ y_5y_9 - y_2y_{10},\ y_7y_8 - y_3y_{10},\ y_6y_9 - y_3y_{10},\ y_7y_9 - y_4y_{10},\\
&{y_8}^2 - y_5y_{10},\ y_8y_9 - y_6y_{10},\ {y_9}^2 - y_7y_{10}\} \ .
\end{aligned} \tag{5.14}$$

Therefore, eq. (5.13) are the 10 primitive invariants of degree 3, obeying 27 syzygies of degree 6 given by eq. (5.14). According to our rule, this should read $10t^3 - 27t^6$. These are precisely the first two terms of eq. (5.12)! Indeed, because we no longer have a complete intersection, the plethystic logarithm of the Poincaré series is not polynomial and continues ad infinitum. What about the 105 and higher terms then, do they mean anything? We will explain this in §5.2.

As a final example of the more subtle case of non-complete-intersection varieties, let us take the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold, with action $(x, y, z) \to (\omega_5 x, \omega_5^2 y, \omega_5^2 z)$. We obtain:

$$\begin{aligned}
M(t; \mathbb{Z}_5) \ &= \ \frac{-1 + t - 3\,t^3 + t^4 - 3\,t^5 + t^7 - t^8}{(-1 + t)^3\,(1 + t + t^2 + t^3 + t^4)^2} = 1 + 3\,t^3 + 2\,t^4 + 7\,t^5 + 5\,t^6 + 4\,t^7 + 11\,t^8\\
&+9\,t^9 + 18\,t^{10} + 15\,t^{11} + 13\,t^{12} + 24\,t^{13} + 21\,t^{14} + 34\,t^{15} + \mathcal{O}(t)^{16} \ ,
\end{aligned} \tag{5.15}$$

giving us

$$PE^{-1}[f_\infty(t)] = 3\,t^3{+}2\,t^4{+}7\,t^5{-}t^6{-}2\,t^7{-}13\,t^8{-}12\,t^9{-}14\,t^{10}{+}14\,t^{11}{+}34\,t^{12}{+}72\,t^{13}{+}47\,t^{14}{+}\mathcal{O}(t)^{15}\ . \tag{5.16}$$

We can find that the 3 invariants of degree 3, 2 of degree 4, 7 of degree 5 are

$$y_{1,\ldots,12} := \{x\,y^2, x\,y\,z, x\,z^2, x^3\,y, x^3\,z, x^5, y^5, y^4\,z, y^3\,z^2, y^2\,z^3, y\,z^4, z^5\}\ . \tag{5.17}$$

We can easily find, using Gröbner algorithms, all relations amongst these 12 invariants, giving us 1 in degree 6 (a quadric in the 3 degree 3 invariants), 2 in degree 7, 13 in degree 8, 12 in deree 9 and 16 in degree 10. All this is in almost in exact agreement with eq. (5.16), with the only exception being that there are 16 degree 10 relations and not 14. Together with the issue of the higher terms in the $\mathbb{C}^3/\mathbb{Z}_3$ case, we now address this discrepancy in the next subsection.

5.2 Plethystics: A Synthetic Approach

We have now witnessed the astounding power of plethystics in the counting problem and have moreover noted a tantalising fact about the geometry of the variety and the (plethystic logarithm of) the generating function for the GIO's in the gauge theory. Let us now attempt to argue why some of the above examples should work. First we note that the Poincaré series f, when finally collected and simplified, is always a rational function. In particular it has a denominator of the form of products of $(1-t^k)$ with possible repeats for k; the numerator is some complicated polynomial. We will call this the **Euler form**. The point is that the coefficient in front of the t^k is always unity and we conjecture that this is a property of the Poincaré series of concern [9].

When we are taking the plethystic logarithm of f, due to the explicit expression eq. (5.5), we are trying to solve the following algebraic problem: find integers b_n such that

$$f(t) = \frac{1}{\prod\limits_{n=1}^{\infty}(1-t^n)^{b_n}}\ , \tag{5.18}$$

where $f(t)$ is a given rational function in Euler form. Note that $PE^{-1}[f(t)] = \sum\limits_{n=1}^{\infty} b_n t^n$, unlike the Poincaré series herself, need not have all positive b_n. Because $f(t)$ has Euler

[9] In fact, all Poincaré series we have encountered, orbifold, toric, etc., have this property. We do not have a rigorous proof of this right now and leave it to the mathematically inclined.

form, the denominator of eq. (5.18) is immediately taken care of. In other words, because $f(t)$ has denominator in the form of products of $(1-t^k)$, all positive values of n and b_n are just read off. These are low values of n and correspond, in the Molien case, to some of the small invariants, including the primitive ones. However, there is still a numerator in $f(t)$, often of complicated form. This will give negative b_n contributions to the RHS of eq. (5.18), which correspond to the relations.

Take $\Delta(27)$ as an example. We need to find b_n such that

$$\frac{1-t^3+t^6}{(1-t^3)^3} = \frac{(1-t^{18})}{(1-t^6)\,(1-t^9)\,(1-t^3)^2} = \frac{1}{\prod_{n=1}^{\infty}(1-t^n)^{b_n}} \ , \tag{5.19}$$

where we have used the identity

$$\frac{(1-t^3)\,(1-t^{18})}{(1-t^6)\,(1-t^9)} = 1-t^3+t^6 \ . \tag{5.20}$$

This rational identity is crucial and expresses even the numerator of f into Euler form. Now we can read out the solution: the denominator contributes terms $+2t^3$, $+t^6$ and $+t^9$ while the numerator contributes $-t^{18}$. Thus $PE^{-1}[M(t)] = 2t^3+t^6+t^9-t^{18}$. In other words, there should be 2 degree 3 invariants, 1 each of degrees 6 and 9, obeying a single relation of degree 18. The fact that the numerator can be factorised into (finite polynomial) Euler form dictates that the plethystic logarithm of f terminates in series expansion as was seen in eq. (5.10). The moral of the story is that

> Findings relations in this language corresponds to finding appropriate factorisations of the numerator into Euler form.

Of course, not all Poincaré series have polynomial plethystic logarithms. This is just the statement that not all polynomials afford identities of the type eq. (5.20). In general, the product on the RHS of eq. (5.18) must be infinite to accommodate those which cannot be put into finite Euler form. These correspond to non-complete intersection varieties. Take $\mathbb{C}^3/\mathbb{Z}_3$, we have

$$M(t;\mathbb{C}_3) = \frac{1+7\,t^3+t^6}{(1-t^3)^3} \ . \tag{5.21}$$

Indeed, no rational identity can express $1+7\,t^3+t^6$ in finite Euler form so the plethystic logarithm will not terminate. Now, as promised earlier, we can explain the higher terms such as the 105 and 540. In this example, there are 10 basic invariants and there are 27 relations amongst them. This explains the first 2 terms in eq. (5.12). This is seen above because if we were to write $1 + 7\,t^3 + t^6$ in Euler form, we would obtain

$$1 + 7\,t^3 + t^6 = \frac{(1-t^6)^{27}(1-t^{12})^{540}\ldots}{(1-t^3)^7(1-t^9)^{105}\ldots} \tag{5.22}$$

thus we get the $+10t^3$ term from the $(1-t^3)^{3+7}$ factor in the denominator and the $-27t^6$ term from the $(1-t^6)^{27}$ factor in the numerator.

Of course, one finds the 27 relations by finding syzygies among the 10 primitive invariants. The reason we can do this is of course Theorem 3.1 which dictates that we need not go beyond degree $|G| = 3$ to find all basic invariants which generate the entire invariant polynomial ring. Instead, if we found the syzygies for the entire invariant ring, we would get the higher terms. That is, we should, considering the expansion of the Molien series

$$\frac{1 + 7\,t^3 + t^6}{(1-t^3)^3} = 1 + 10\,t^3 + 28\,t^6 + 55\,t^9 + 91\,t^{12} + 136\,t^{15} + 190\,t^{18} + \ldots\ , \tag{5.23}$$

consider all $10 + 28 + 55 + \ldots$ invariants as polynomials in 3 variables and find all their syzygies; this should give the higher terms. Of course this cannot be done all at once, but nevertheless we can consider the process stepwise: first, syzygies for 10 of them, then $10 + 28$ of them, etc. In principle, if we only wish to know up to some degree, we only need to find syzygies for invariants up to that degree. This algorithm is the precise analogue of the infinite product expansion of eq. (5.22) into Euler form, which serves as successive approximation to the LHS in eq. (5.22). This also explains the discrepancies in the case of $\mathbb{C}^3/\mathbb{Z}_5$ as seen above. In these cases where the Euler product is non-terminating, and rational identities become infinite products, the syzygies should thereby receive stepwise corrections. We should be able to arrive at the right answer after some finite number of steps if we only wish to know the terms up to a desired order.

Let us check up to second order in this example of $\mathbb{C}^3/\mathbb{Z}_3$ by finding the syzygies amongst the 10 basic invariants of degree 3 and 28 degree 6 invariants. We find, using [42], 595 relations: 55 of degree 6, 225 of degree 9 and 315 of degree 12. This thus reads

$$10t^3 + 28t^6 - 55t^6 - 225t^9 - 315t^{12} = 10t^3 - 27t^6 - 225t^9 - 315t^{12}\ . \tag{5.24}$$

Good, we reproduce the first 2 terms of $PE^{-1}[M(t)]$ and have the next 2 terms. This is only up to order 2, i.e., finding syzygies among 38 polynomials. At next order, we would have to find relations among $10+28+55=93$ polynomials and correct the t^9 and t^{12} coefficients; the computation becomes increasingly strenuous for the computer[10].

5.3 Complete Intersections

We see from the preceeding arguments that the most powerful avatar of the intimate relations between plethystics and syzygies is realised in complete intersections, especially in single hypersurfaces. We have seen that $\Delta(27)$ is one such example of the hypersurface. The key feature for this class of varieties is that *the series for* $f_1 = PE^{-1}[f_\infty(t)]$ *terminates and is polynomial*. This is nice because if we knew the defining equations and the degrees of the various pieces, we could re-construct the Poincaré series and find the number of invariants in the gauge theory! This is independent of whether the variety is orbifold or toric, but should hold in general. In fact, we do not even need to know what the gauge theory is! We shall see, in §6.2, an inverse application of this philosophy, where we shall construct a variety with desired gauge invariants.

5.3.1 del Pezzo Family Revisited

Take a non-orbifold, non-toric, single hypersurface, the famous cubic in $\mathbb{P}^3$; this is the cone over the 6-th del Pezzo surface. From eq. (3.27) and eq. (5.7), we have

$$f(t;dP_6) = \frac{1+t+t^2}{(1-t)^3} \Rightarrow PE^{-1}[f(t;dP_8)] = 4t - t^3 \ , \tag{5.25}$$

which says that there should be 4 linear invariants, obeying 1 cubic relation; precisely the definition of dP_6.

Another illustrative example is dP_8; here we shall go beyond projective spaces, but rather to weighted projective spaces. We shall see that the plethystic logarithm still works. We know (cf. [55]) that dP_8 as a surface is given by a single equation in $W\mathbb{P}^3_{1,1,2,3}$. Again, from eq. (3.27) and eq. (5.7), we find

$$f(t;dP_8) = \frac{1-t+t^2}{(1-t)^3} \Rightarrow PE^{-1}[f(t;dP_8)] = 2\,t + t^2 + t^3 - t^6 \ . \tag{5.26}$$

This is again correct: 2 degree 1, 1 degree 2 and 1 degree 3, obeying a single degree 6 relation. This can only happen in a weighted projective space, viz., $W\mathbb{P}^3_{1,1,2,3}$. Thus, our

[10] This alternative addition of invariants and substraction of relations is reminiscent of the characters of minimal models and the removal of null-states using the Kac determinant in 2-dimensional conformal field theories.

proposal eq. (3.27) is again confirmed. We note that, upon comparing eq. (5.26) and the result eq. (5.10) for $\Delta(27)$, the f-functions are the same, only with the replacement $t \to t^3$. This does not surprise us, indeed (cf. e.g. [7, 56, 57]) $\Delta(27)$ is known to be a special point in the moduli space of dP_8's.

In fact, all del Pezzo surfaces for $n > 4$ are complete intersections (cf. e.g., eq.(3.2) of [55] and also [58]). We check against eq. (3.27) and find complete agreement. For clarity, let us tabulate these results:

dP_n	$f = f_\infty(t)$	$f_1 = PE^{-1}[f(t)]$	Defining Equations
5	$\frac{1+2t+t^2}{(1-t)^3}$	$5\,t - 2\,t^2$	2 degree 2 equations in $\mathbb{P}^4$
6	$\frac{1+t+t^2}{(1-t)^3}$	$4\,t - t^3$	1 degree 3 equation in $\mathbb{P}^3$
7	$\frac{1+t^2}{(1-t)^3}$	$3\,t + t^2 - t^4$	1 degree 4 equation in $W\mathbb{P}^3_{1,1,1,2}$
8	$\frac{1-t+t^2}{(1-t)^3}$	$2\,t + t^2 + t^3 - t^6$	1 degree 6 equation in $W\mathbb{P}^3_{1,1,2,3}$

(5.27)

Therefore, for the entire del Pezzo family, members 0 to 3 are checked by toric methods while 5 to 8 are complete intersections. The only one remaining is dP_4 and from eq. (3.27),

$$f(t; dP_4) = \frac{1 + 3\,t + t^2}{(1-t)^3} = 1+6\,t+16\,t^2+31\,t^3+51\,t^4+76\,t^5+106\,t^6+141\,t^7+181\,t^8+226\,t^9+\mathcal{O}(t^{10}) \tag{5.28}$$

predicts the single-trace GIO counting for this variety. The equation for this variety [51, 58] is the (non-complete) intersection of 5 quardrics in $\mathbb{P}^5$ (cf. also eq. 5.29 of [27]). Expanding the plethystic logarithm of f in this case gives

$$PE^{-1}[f(t; dP_4)] = 6\,t - 5\,t^2 + 5\,t^3 - 10\,t^4 + 24\,t^5 - 55\,t^6 + 120\,t^7 - \mathcal{O}(t^8) \tag{5.29}$$

We see that the first 2 terms are actually correct: there are 5 degree 2 relations in 6 variables!

For full reference, we tabulate below the other members of the del Pezzo family, these are non-complete intersections:

dP_n	$f = f_\infty(t)$	$f_1 = PE^{-1}[f(t)]$	Defining Equations
0	$\frac{1+7t+t^2}{(1-t)^3}$	$10\,t - 27\,t^2 + 105\,t^3 - 540\,t^4 + 3024\,t^5 + \mathcal{O}(t)^6$	$(10\|2^{27})$
1	$\frac{1+6t+t^2}{(1-t)^3}$	$9\,t - 20\,t^2 + 64\,t^3 - 280\,t^4 + 1344\,t^5 + \mathcal{O}(t)^6$	$(9\|2^{20})$
2	$\frac{1+5t+t^2}{(1-t)^3}$	$8\,t - 14\,t^2 + 35\,t^3 - 126\,t^4 + 504\,t^5 + \mathcal{O}(t)^6$	$(8\|2^{14})$
3	$\frac{1+4t+t^2}{(1-t)^3}$	$7\,t - 9\,t^2 + 16\,t^3 - 45\,t^4 + 144\,t^5 + \mathcal{O}(t)^6$	$(7\|2^{9})$
4	$\frac{1+3t+t^2}{(1-t)^3}$	$6\,t - 5\,t^2 + 5\,t^3 - 10\,t^4 + 24\,t^5 + \mathcal{O}(t)^6$	$(6\|2^{5})$

(5.30)

Here, we have computed these defining equations using fat-point methods on $\mathbb{P}^2$ [42, 59]. We have used the notation, in the above table, that $(m|p_1^{q_1} \ldots p_k^{q_k})$ denotes q_1 equations of degree p_1, q_2 equations of degree p_2, etc., all in m variables. The first memeber, dP_0, is of course $\mathbb{C}^3/\mathbb{Z}_3$ as studied in detail in eq. (5.13) and eq. (5.14). Furthermore, as mentioned when we first presented eq. (3.27), F_0 has the same $f(t)$ as dP_1. Indeed, we can study the degree 2 Veronese-Segrè embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^8$ and see that F_0 also has defining equation in 9 variables as $(9|2^{20})$. The precise forms of these equations, of course, differ from those of dP_1. Thus, dP_1 and F_0 are in different points of a complex structure moduli space.

To compare and contrast, we include the f_1 results for the ADE-series addressed in §3.1.1; indeed these are all complete intersections - in fact, single hypersurfaces - so f_1, the plethystic logarithm of the the Molien series should be polynomial:

$G \subset SU(2)$	$f_1 = PE^{-1}[M(t;\ G)]$	Defining Equation in $\mathbb{C}[u,v,w]$
$\hat{A}_{n-1}$	$t^2 + 2t^n - t^{2n}$	$uv = w^n$
$\hat{D}_{n+2}$	$t^4 + t^{2n} + t^{2n+2} - t^{4n+4}$	$u^2 + v^2 w = w^{n+1}$
$\hat{E}_6$	$t^6 + t^8 + t^{12} - t^{24}$	$u^2 + v^3 + w^4 = 0$
$\hat{E}_7$	$t^8 + t^{12} + t^{18} - t^{36}$	$u^2 + v^3 + vw^3 = 0$
$\hat{E}_8$	$t^{12} + t^{20} + t^{30} - t^{60}$	$u^2 + v^3 + w^5 = 0$

(5.31)

5.3.2 Example: The Hypersurface $x^2 + y^2 + z^2 + w^k = 0$

Now, let us try another family of complete intersection 3-folds, viz., $x^2+y^2+z^2+w^k = 0$ in $\mathbb{C}^4$. For $k = 1$, this is just $\mathbb{C}^3$, for $k = 2$, it is the conifold $\mathcal{C}$. For $k > 2$, the theory is studied in [14]. However, for $k \geq 3$, [60, 61] recently showed that there is no Sasaki-Einstein metric, whereby making the AdS/CFT correspondence a little ambiguous here.

For $k = 2n$ even, we have x, y, z being degree n and w being degree 1. From this we can read out $f_1 = t + 3t^n - t^{2n}$. Thus we can calculate that

$$f_\infty(k = 2n) = PE[t + 3t^n - t^{2n}] = \frac{(1 - t^{2n})}{(1 - t)(1 - t^n)^3} \ .$$

Similarly, for k odd, we have x, y, z being degree k and w, degree 2. From this we can read out that $f_1 = t^2 + 3t^k - t^{2k}$ and whence

$$f_\infty(k) = PE[t^2 + 3t^k - t^{2k}] = \frac{(1 - t^{2k})}{(1 - t^2)(1 - t^k)^3}, \qquad , k \text{ odd} \ .$$

To demonstrate, we list the series expansion of the Poincaré series f_∞ for $k = 1$ to $k = 5$ and we find

$$\begin{array}{rcl}
f_\infty(1) & = & f_\infty(t;\ \mathbb{C}^3) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 + 36t^7 + ... \\
f_\infty(2) & = & f_\infty(t;\ \mathcal{C}) = 1 + 4t + 9t^2 + 16t^3 + 25t^4 + 36t^5 + 49t^6 + 64t^7 + ... \\
f_\infty(3) & = & 1 + t^2 + 3t^3 + t^4 + 3t^5 + 6t^6 + 3t^7 + 6t^8 + 10t^9 + 6t^{10} + ... \\
f_\infty(4) & = & 1 + t + 4t^2 + 4t^3 + 9t^4 + 9t^5 + 16t^6 + 16t^7 + 25t^8 + 25t^9 + 36t^{10} + 36t^{11} + ... \\
f_\infty(5) & = & 1 + t^2 + t^4 + 3t^5 + t^6 + 3t^7 + t^8 + 3t^9 + 6t^{10} + 3t^{11} + 6t^{12} + 3t^{13} + 6t^{14} + ...
\end{array}$$

5.4 Refined Relations: The Conifold Revised

In all of the above, we have used the generating function with a single variable t. How does this all work if everything is refined fully so as to contain a tuple of dummy variables for the various $U(1)$-charges? We shall now see that the relations are still explicitly encoded by f_1. We mentioned early on that the F-term relations are automatically built into the counting. Indeed, for $\mathbb{C}^3$, we do not have these relations - just that x, y, z commute. For the conifold, we have the simplest demonstration that f_1 contains relations.

Recall the expression for the Poincaré series f_∞ in eq. (4.6). Now, let us take the multi-variate plethystic logarithm [53], which for $f_\infty(t_1, \ldots, t_m)$ is, recollecting eq. (5.6)

$$PE^{-1}[f_\infty(t_1, \ldots, t_m)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(f_\infty(t_1^k, \ldots, t_m^k)) \ . \qquad (5.32)$$

The result is

$$f_1 = PE^{-1}[\frac{xy(1 - q^2)}{(1 - qx)(1 - qy)(q - x)(q - y)}] = \frac{q}{x} + q\,x + \frac{q}{y} + q\,y - q^2 \ . \qquad (5.33)$$

Indeed, f_1 is polynomial because the conifold is complete intersection. There are four invariants, corresponding to $qx, \frac{q}{x}, qy, \frac{q}{y}$. If we would write just these generators without any subtractions, this would be merely the result for $\mathbb{C}^4$. Therefore it is not enough. To put the relation, we notice that $(qx)(q/x) = (qy)(q/y) = q^2$ and therefore we should subtract one combination of q^2. We thus reproduce eq. (5.33). The procedure is simple and analogous for complete intersections. However, for non-complete intersections, once we make the subtraction we are taking away too much. We must therefore compensate by adding those which are subtracted, etc. ad infinitum, just like the non-terminating series explained in §5.2.

6 Asymptotics and the Meinardus Theorem

We have encountered, in the preceeding discussions, many infinite products of Euler type. Indeed, we recall that the generating function for multi-trace GIO's is

$$g(t) := \sum_{n=0}^{\infty} p_n t^n = PE[f(t)] = \frac{1}{\prod\limits_{n=1}^{\infty} (1-t^n)^{a_n}} \qquad \text{for} \qquad f(t) = \sum_{m=0}^{\infty} a_m t^m \ . \tag{6.1}$$

In the case that all $a_n = 1$, $g(t)$ is the Euler function, or, up to a factor of $t^{-\frac{1}{24}}$, the Dedekind η-function. This is our familiar generating function for the number of ways of partitioning integers. The Hardy-Ramanujan equation gives the asymptotic behaviour of p_n and was what gave rise to the Hagedorn temperature (q.v. [64]). It is, needless to say, important to find analogous asymptotic behaviours for general a_n. This would give micro-state counting for our quiver gauge theories.

Luckily, this generalisation of Hardy-Ramanujan is known. This is a result due to G. Meinardus [62] (q.v. [63], to whose notation we adhere, for some explicit results). Meinardus' theorem states that the asymptotic behaviour of p_n in eq. (6.1) is:

$$p_n \sim C_1 n^{C_2} \exp\left[n^{\frac{\alpha}{\alpha+1}} (1+\frac{1}{\alpha}) \left(A \Gamma(\alpha+1) \zeta(\alpha+1) \right)^{\frac{1}{\alpha+1}} \right] (1 + \mathcal{O}(n^{-C_3})) \ , \tag{6.2}$$

if the Dirichlet series for the coefficients a_m of f, defined as

$$D(s) := \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \qquad \mathrm{Re}(s) > \alpha > 0 \ , \tag{6.3}$$

converges and is analytically continuable into the strip $-C_0 < \mathrm{Re}(s) \le \alpha$ for some real constant $0 < C_0 < 1$ and such that in this strip, $D(s)$ has only 1 simple pole at

$s = \alpha \in \mathbb{R}_+$ with residue A. The constants in eq. (6.2) are

$$\begin{aligned} C_1 &= e^{D'(0)} \frac{1}{\sqrt{2\pi(\alpha+1)}} \left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1-2D(0)}{2(\alpha+1)}} , \\ C_2 &= \frac{D(0)-1-\frac{\alpha}{2}}{\alpha+1} , \end{aligned} \tag{6.4}$$

and C_3 some positive constant.

6.1 Example: $\mathbb{C}$ and Dedekind η

For example, when all $a_m = 1$, we have the usual partition of integers and the Dirichlet series is just the Riemann ζ-function. The generating function $f(t) = \sum\limits_{m=0}^{\infty} t^m$ is of course simply $\frac{1}{1-t}$ and the geometry is that of the complex line $\mathbb{C}$. Using the above results of Meinardus, we have

$$\alpha = 1, \quad A = 1, \quad D(0) = -\frac{1}{2}, \quad D'(0) = e^{-\frac{1}{2}\log(2\pi)}; \qquad C_1 = \frac{1}{4\sqrt{3}}, \quad C_2 = -1 ,$$

giving us

$$p_n \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} (1 + \mathcal{O}(n^{-C})) ,$$

precisely the Hardy-Ramanujan result.

6.2 Example: MacMahon Function and a Riemann Surface

Next, consider

$$f(t) = \frac{1-t+t^2}{(1-t)^2}, \; a_n = n, \quad \Rightarrow \quad g(t) = \frac{1}{\prod\limits_{n=1}^{\infty}(1-t^n)^n} .$$

As $PE^{-1}[f(t)] = t + t^2 + t^3 - t^6$, this is a complete intersection, given as a hypersurface of degree 6 in $W\mathbb{P}^2_{1,2,3}$. The dimension is therefore 1 and is hence a Riemann surface. The total space is an affine cone over this surface and is of dimension 2. We can embed $W\mathbb{P}^2_{1,2,3}$, through a Veronese-Segré map, into $\mathbb{P}^6$ and see that the genus of the Riemann surface is 1. Alternatively, we can projectivise the weight-one coordinate in $W\mathbb{P}^2_{1,2,3}$ and simply obtain a standard elliptic curve. Therefore, the geometry is a cone over a torus! The generating function $g(t)$ is the well-known MacMahon function [65]. We see that

$$\alpha = 2, \quad A = 1, \quad D(0) = -\frac{1}{12}, \quad D'(0) = \frac{1}{12} - \log(G_l); \qquad C_1 = \frac{e^{\frac{1}{12}}\,\zeta(3)^{\frac{7}{36}}}{2^{\frac{11}{36}}\,G_l\,\sqrt{3}\,\pi}, \quad C_2 = -\frac{25}{36} ,$$

where $G_l := e^{\frac{1}{12} - \zeta'(-1)} \simeq 1.28243$ is the Glaisher constant. Hence,

$$p_n \sim \frac{e^{\frac{1}{12}}\, \zeta(3)^{\frac{7}{36}}}{2^{\frac{11}{36}}\, G_l \sqrt{3\,\pi}} n^{-\frac{25}{36}} \exp\left(\frac{3\,\zeta(3)^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}} \right) \ . \tag{6.5}$$

6.3 Example: Our Familiar $\mathbb{C}^3$

Returning to something we have encountered earlier, let us attack the $\mathbb{C}^3$ example of eq. (2.11). We can now find the coefficients d_k therein! Using eq. (5.5) we have that

$$g(t;\ \mathbb{C}^3) = PE[f(t;\ \mathbb{C}^3)] = \frac{1}{\prod\limits_{n=1}^{\infty} (1-t^n)^{a_n}}, \qquad a_n = \frac{1}{2}(n+1)(n+2) \ . \tag{6.6}$$

We can readily see that $D(s) = \frac{1}{2}\left(\zeta(-2+s) + 3\zeta(-1+s) + 2\zeta(s)\right)$. We see that there are 3 poles, at 1,2 and 3. Of course, Meinardus Theorem requires that there be only one pole within a strip. Thus, one must consider one monomial of a_n at a time and consider the break-down

$$g(t;\ \mathbb{C}^3) = \frac{1}{\prod\limits_{n=1}^{\infty} (1-t^n)^{a_{n_1}}} \cdot \frac{1}{\prod\limits_{n=1}^{\infty} (1-t^n)^{a_{n_2}}} \cdot \frac{1}{\prod\limits_{n=1}^{\infty} (1-t^n)^{a_{n_3}}} := g_1(t) g_2(t) g_3(t),$$

with $a_{n_1} = \frac{1}{2}n^2$, $a_{n_2} = \frac{3}{2}n$, $a_{n_3} = 1$. Applying Meinardus and defining $g_{i=1,2,3} := \sum\limits_{n=0}^{\infty} p_i(n) t^n$, we have that

$$\begin{aligned} &p_1(n) \sim \frac{e^{\frac{\zeta'(-2)}{2}}}{2\,2^{\frac{5}{8}}\,15^{\frac{1}{8}}} n^{-\frac{5}{8}} \exp\left(\frac{2\,2^{\frac{3}{4}}\,\pi}{3\,15^{\frac{1}{4}}} n^{\frac{3}{4}} \right), \quad p_2(n) \sim \frac{e^{\frac{1}{8}}\,\zeta(3)^{\frac{5}{24}}}{3^{\frac{7}{24}}\,G_l^{\frac{3}{2}}\,\sqrt{2\,\pi}} n^{-\frac{17}{24}} \exp\left(\frac{3\,(3\,\zeta(3))^{\frac{1}{3}}}{2} n^{\frac{2}{3}} \right), \\ &p_3(n) \sim \frac{1}{4 \cdot 3^{\frac{3}{8}}} n^{-\frac{7}{8}} \exp\left(\frac{\pi}{\sqrt{3}} n^{\frac{1}{2}} \right) \ . \end{aligned} \tag{6.7}$$

Therefore, we have the convolution $p(n) = \sum\limits_{r+s+t=n} p_1(r) p_2(s) p_3(t)$ and since the exponential growth of $p_1(n)$ dominates over the other two, for large n

$$p(n) \sim p_1(n) \ . \tag{6.8}$$

7 Single-Trace and Multi-Trace for Finite N

We have, in all preceeding discussions, made the important simplification of taking N, the matrix size of the operators, to infinity, whereby decoupling spurious relations which arise from the lack of commutativity among the matrices of finite size. As mentioned in the introduction, the problem of counting for finite N is a significantly more difficult one. Nevertheless, we shall see in this section that the plethystics are still applicable.

We consider the problem of counting BPS states of $\mathcal{N} = 1$ supersymmetric quiver gauge theories for N finite; N is the number of D3-branes at the tip of the CY cone. We denote the generating function for multi-trace GIO's by g_N. This problem is of significant interest, for instance, to studying phase transitions and AdS_5 black holes. We already considered in the previous sections the functions $g_1 = f_\infty$ and $g_\infty = PE[g_1]$, and we are going to propose that it is still quite simple to reconstruct g_N in terms of g_1.

Suppose the single-trace generating function is given by $g_1(t) = f_\infty(t) = \sum\limits_{n=0}^{\infty} a_n t^n$, then we can construct the following function:

$$g(\nu; t) := \prod_{m=0}^{\infty} \frac{1}{(1 - \nu\, t^m)^{a_m}} \ . \tag{7.1}$$

We immediately notice a strong similarity to the Euler product form of the plethystic exponential introduced in eq. (5.1), eq. (5.5) and especially eq. (5.6).

We now propose that the finite N multi-trace generating function $g_N(t)$ is simply given by the expansion

$$\sum_{N=0}^{\infty} g_N(t) \nu^N = g(\nu; t) \ . \tag{7.2}$$

We have 2 limiting cases to check, viz., g_1 and g_∞, with which we are now quite familiar. First, we note that

$$\partial_\nu g(\nu; t) = \sum_{k=0}^{\infty} \frac{(-a_k)(-t^k)}{(1 - \nu\, t^k)^{a_k+1}} \prod_{m=0; m \neq k}^{\infty} \frac{1}{(1 - \nu\, t^m)^{a_m}} = g(\nu; t) \sum_{k=0}^{\infty} \frac{a_k t^k}{(1 - \nu\, t^k)} \ .$$

Furthermore, since $g(0; t) = 1$, we have that $\partial_\nu g(0; t) = \sum_{k=0}^{\infty} a_k t^k$. Therefore, the coefficient of ν in eq. (7.2) is indeed our g_1:

$$\partial_\nu g(\nu, q)|_{\nu=0} = g_1(q) \ . \tag{7.3}$$

Next, let us check whether the N-th coefficient for $N \to \infty$ gives our g_∞. This coefficient can be found by considering the limit[11] $\lim\limits_{\nu \to 1} (1 - \nu)^{a_0} g(\nu; t)$ which extracts the

[11] In this paper we always have $a_0 = 1$, since the only operator of vanishing scaling dimension is the identity. We can see this explicitly in all the examples we have given throughout.

large N-term in the series expansion. We see that

$$\lim_{\nu\to 1}(1-\nu)^{a_0} g(\nu;t) = \prod_{m=1}^{\infty} \frac{1}{(1-\ t^m)^{a_m}} \equiv PE[g_1(t)] \ . \tag{7.4}$$

Therefore, our expansion eq. (7.2) has the property that its large N coefficient is the PE of the linear coefficient, precisely what is required of g_∞. We will see in the ensuing text why the expansion does what it is supposed to.

It is very interesting to compare eq. (5.6) and eq. (7.1). From it we can see that the parameter ν in eq. (7.1) can have two different interpretations:

I. It counts the number of single-trace GIO's in a multi-trace GIO for the limit of matrix rank $N \to \infty$ (here we include the single trace of identity as well) as in eq. (5.6);

II. It counts the number of multi-trace GIO's for matrix rank N given by the finite power of ν as in eq. (7.2).

Naively these two counting problems seem to be unrelated, but our proposed formula eq. (7.1) indicates that they are the same.

Formulae (7.1) and (7.2) give the general solution for counting multi-trace BPS GIO's, for a finite number N of D3-branes. In fact, the relation between f_N and g_N, in general, still obeys the plethytic exponential as was in eq. (2.8) and eq. (5.1), which we summarise now (for a list of variables t_i):

$$g_N(t_i) = PE[f_N(t_i)] = \exp\left(\sum_{k=1}^{\infty} \frac{f_N(t_i^k) - f_N(0,\ldots,0)}{k}\right) \ . \tag{7.5}$$

Symmetric Products and Moduli Spaces: We can in fact re-examine the finite N counting from another perspective. The standard lore for N D3-branes probing a CY manifold, X, is that the moduli space of vacua, $\mathcal{M}_{\rm vac}(N;X)$, is the symmetric product of N copies of the CY manifold,

$$\mathcal{M}_{\rm vac}(N;X) = \mathcal{S}^N(X) := \frac{X^N}{S_N}, \tag{7.6}$$

where S_N is the permutation group of N elements. Following our general line in this paper we can now state two important relations:

1. g_N counts multi trace operators for one D3-brane on the symmetric product of N CY manifolds $\mathcal{M}_{\text{vac}}(N; X)$:

$$g_N(t; X) = g_1(t; \frac{X^N}{S_N}) = f_\infty(t; \frac{X^N}{S_N}). \tag{7.7}$$

   Alternatively we can think of it as the Poincaré series for $\mathcal{M}_{\text{vac}}(N; X)$. Furthermore, from the second equality we conclude that g_N also counts the single trace operators on $\mathcal{M}_{\text{vac}}(N; X)$ in the limit in which there are no matrix relations at all, $N \to \infty$.

2. f_N counts single trace operators for one D3-brane on the symmetric product of N CY manifolds $\mathcal{M}_{\text{vac}}(N; X)$:

$$f_N(t; X) = f_1(t; \frac{X^N}{S_N}). \tag{7.8}$$

   Here, we are again using the plethystic exponential relations. In fact, in cases in which the symmetric product is a complete intersection, f_N will be finite and we can compare the computation of f_N using the formulas at the beginning of this section to independent derivations using the property that the manifold is a complete intersection. In cases in which the symmetric product is not a complete intersection we can still use the reasoning of §5.2. To count the number of generators and the number of defining relations for the symmetric product.

Actually, eq. (7.7) is the reason why eq. (7.2) works. The far-LHS of the expression is the generating function for multi-trace at finite N, in line with interpretation II stated above, while the far-RHS is the single-trace generating function at $N \to \infty$, in accord with interpretation I. Indeed, multi-trace operators with N single-trace components in X is in one-to-one correspondence with single-trace operators in $Sym^N(X)$. Therefore, eq. (7.7) serves to bridge the two, whereby showing that the expansion coefficients g_N indeed count multi-trace operators at finite N. Having delved into much abstraction, let us be concrete and now show how these proposals agree with known results.

7.1 Example: The Complex Line $\mathbb{C}$

The simplest example, as was encountered in §6.1, is given by

$$g_1(t) = f_\infty(t;\ \mathbb{C}) = \frac{1}{(1-t)} = \sum_{n=0}^{\infty} t^n\ . \tag{7.9}$$

This is the well known partition function of the half-BPS states in $\mathcal{N}=4$ SYM (given a particular choice of the supercharges). This partition function also counts the "extremal" BPS mesons in toric quivers (i.e. the mesons lying along an edge of the toric cone). In this case, it should be simple to check the multi-trace generating function is, as dictated by eq. (7.1) and eq. (7.2), given precisely by

$$g(\nu;t)=\sum_{N=0}^{\infty} g_N(t)\nu^N=\prod_{m=0}^{\infty}\frac{1}{(1-\nu t^m)}\ . \tag{7.10}$$

We note that g_N is also the partition function of N bosonic one-dimensional harmonic oscillators. In other words, we are taking a quantum particle whose single particle states are precisely the integer points in the half-line $\mathbb{Z}_{\geq 0}$, and considering the placement of N of such bosons. We can obtain g_N for any value of N by Taylor expansion:

$$g_N(t)=\prod_{n=1}^{N}\frac{1}{1-t^n}. \tag{7.11}$$

In fact, there is another way to see eq. (7.11). Indeed, we have the single-trace generating function explicitly:

$$f_N(t;\ \mathbb{C})=1+t+t^2+...+t^N=\frac{1-t^{N+1}}{1-t}\ , \tag{7.12}$$

which encode the operators $\mathrm{Tr}(X^i)$ for $i=0,\dots,N$. We can take the PE of eq. (7.12) and using the multiplicative property eq. (5.5), arrive at eq. (7.11) directly.

A few specific cases are at hand. For $N=1$,

$$g_1(t)=\frac{1}{1-t}=1+t+t^2+\ldots+t^n+\ldots\ , \tag{7.13}$$

corresponding to the operators

$$\mathbb{I};\ \mathrm{Tr}(X);\ \mathrm{Tr}(X)^2;\ \ldots;\ \mathrm{Tr}(X)^n;\ \ldots\ .$$

For $N=2$ we get

$$g_2(t)=\frac{1}{(1-t)(1-t^2)}=1+t+2t^2+2t^3+3t^4+3t^5+4t^6+\ldots+(n+1)t^{2n}+(n+1)t^{2n+1}+\ldots\ , \tag{7.14}$$

corresponding to the operators (we have dropped the Tr in the notation without ambiguity):

$$\begin{array}{l}\mathbb{I};\qquad (X);\qquad (X)^2,(X^2);\qquad (X)^3,(X)(X^2);\qquad (X)^4,(X)^2(X^2),(X^2)^2;\ \ldots\\ \ldots;\ (X)^{2n},(X^2)(X)^{2n-2},(X^2)^2(X)^{2n-4},\ldots,(X^2)^{n-1};\ldots\ .\end{array}$$

Indeed $g_2(t)$ is the Poincaré series for $\mathbb{C}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by the exchange of the two coordinates (z_1, z_2) of $\mathbb{C}^2$. It has two generators, one of degree 1 corresponding to $a := z_1 + z_2$ and another of degree 2, corresponding to $b := z_1 z_2$. It is easy to see that the other invariant of degree 2 is represented in terms of these two, $z_1^2 + z_2^2 = a^2 - 2b$. Similarly, all other invariants of higher degree can be written in terms of these two.

For $N = 3$, we have

$$g_3(t) = \frac{1}{(1-t)(1-t^2)(1-t^3)} = 1+t+2t^2+3t^3+4t^4+5t^5+7t^6+8t^7+10t^8+12t^9+14t^{10}+\mathcal{O}(t^{11}) \ , \tag{7.15}$$

corresponding to the operators

$$\mathbb{I}; \quad (X); \quad (X^2), (X)^2; \quad (X^3), (X)(X^2), (X)^3; \quad (X)(X^3), (X^2)^2, (X)^2(X^2), (X)^4;$$
$$(X^2)(X^3), (X)^2(X^3), (X)(X^2)^2, (X)^3(X^2), (X)^5; \ldots \ .$$

We see that indeed our generating function eq. (7.2) agrees with the explicit counting.

To demonstrate the interplay between plethystics and symmetric products, we now calculate f_N using eq. (7.8). We need to find $f_N(t;\ \mathbb{C}) = f_1(t; \frac{\mathbb{C}^N}{S_N})$. Now, if we expand an N-th order polynomial equation in one complex variable x,

$$P_N(x) = x^N + \sum_{i=1}^{N} a_i x^{N-i} = \prod_{j=1}^{N} (x - z_j), \tag{7.16}$$

we find that the parameters a_i, $i = 1 \ldots N$ are symmetric functions of degree i for the coordinates z_j on $\mathbb{C}^N$. That is, a_i are coordinates on $\mathcal{M}_{\text{vac}}(N; \mathbb{C}) = \mathbb{C}^N/S_N$. Furthermore, there is precisely one generator of degree i for the ring of symmetric functions of the z_j for any i between 0 and N. We can pick the generators to be the coordinates a_i. Any other symmetric function of degree $i > N$ can be written in terms of the a_i. Collecting this together we find f_1 for $\mathcal{M}_{\text{vac}}(N; \mathbb{C})$ as in eq. (7.12),

$$f_1(t; \frac{\mathbb{C}^N}{S_N}) = \frac{1 - t^{N+1}}{1-t} = f_N(t; \mathbb{C}), \tag{7.17}$$

consistent with our proposal from eq. (7.8) which implies, using the plethystic exponential, eq. (7.7), thus supporting the proposal for multi-trace.

7.2 Example: The Complex Plane $\mathbb{C}^2$

Next, we address a slightly more involved example, viz., $\mathbb{C}^2$. This case is quite simple as well and describes 1/4-BPS operators in $U(N)$ $\mathcal{N} = 4$ SYM. This also describes a subsector of BPS operators in many toric quivers, namely the operators corresponding to points lying along a face of the toric cone (this face gives a toric subcone of the toric cone, a cone over a SUSY 3 cycle), when the SUSY 3-cycle has the topology of S^3. Now, we have that

$$\begin{aligned}
f_1(t;\ \mathbb{C}^2) &= 2t; \\
g_1(t;\ \mathbb{C}^2) &= PE[f_1(t;\ \mathbb{C}^2)] = \frac{1}{(1-t)^2} = \sum_{n=0}^{\infty}(n+1)t^n\ .
\end{aligned} \tag{7.18}$$

Formulae (5.6) and (7.1) takes the form

$$g(\nu;t) = \prod_{m=0}^{\infty} \frac{1}{(1-\nu t^m)^{m+1}} = \exp(\sum_{k=0}^{\infty} \frac{\nu^k}{k(1-t^{2k})}) \tag{7.19}$$

which looks deceptively similar to the generalized MacMahon function which is used as the partition function for the topological string on the conifold [66]. Using eq. (7.2), we get

$$g_2(t) = \frac{1+t^2}{(1-t)^4(1+t)^2} = 1 + 2t + 6t^2 + 10t^3 + 19t^4 + 28t^5 + 44t^6 + \ldots\ . \tag{7.20}$$

We report the R-charge 3 GIO's, corresponding to the term $10t^3$. We see that indeed there are $10 = 2+3+3+2$ of them:

$$(X^2)(X), (X)^3; \qquad (X^2)(Y), (X)(XY), (X)^2(Y); \qquad [X \leftrightarrow Y]\ .$$

Next, for R-charge 4 GIO's, we see that there are indeed $19 = 3+4+5+4+3$ them:

$$\begin{gathered}
(X^2)^2, (X^2)(X)^2, (X)^4; \\
(X^2)(X)(Y), \quad (X)^2(XY), \quad (X^2)(XY), \quad (X)^3(Y); \\
(X^2)(Y^2), \quad (X^2)(Y)^2, \quad (X)^2(Y^2), \quad (X)(XY)(Y), \quad (X)^2(Y)^2; \\
[X \leftrightarrow Y]
\end{gathered}$$

The moduli space of vacua for this case is $(\mathbb{C}^2)^2/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts as exchange of the two coordinates. It is a complete intersection and has

$$f_1(t;(\mathbb{C}^2)^2/\mathbb{Z}_2) = f_2(t;\mathbb{C}^2) = 1 + 2t + 3t^2 - t^4. \tag{7.21}$$

7.3 Example: The Conifold

The single trace GIO's for the conifold are given, recalling eq. (4.5), by

$$g_1(t) = \frac{1+t}{(1-t)^3} = \sum_{n=0}^{\infty}(n+1)^2 t^n \ . \tag{7.22}$$

From formula (7.1) we get

$$f_2(t) = \frac{1+t+7\,t^2+3\,t^3+4\,t^4}{(1-t)^3\,(1-t^2)^3} = 1+4t+19t^2+52t^3+134t^4+280t^5+554t^6+\ldots\ , \tag{7.23}$$

corresponding to the operators (again, we drop the Tr for brevity):

$$\mathbb{I}; \qquad (M_{i,j}); \ldots \ . \tag{7.24}$$

At R-charge 2 we have 9 single-trace GIO's (cf. Fig. 4) and 10 double-trace GIO's, given explicitly by:

$$\begin{array}{ccc} (M_{0,1}M_{0,1}) & (M_{0,1}M_{1,0}) & (M_{1,0}M_{1,0}) \\ (M_{0,1}M_{-1,0}) & (M_{0,1}M_{0,-1}) = \ (M_{1,0}M_{-1,0}) & (M_{1,0}M_{0,-1}) \\ (M_{-1,0}M_{-1,0}) & (M_{-1,0}M_{0,-1}) & (M_{0,-1}M_{0,-1}) \ . \end{array} \tag{7.25}$$

and

$$\begin{array}{ccc} (M_{0,1})(M_{0,1}) & (M_{0,1})(M_{1,0}) & (M_{1,0})(M_{1,0}) \\ (M_{0,1})(M_{-1,0}) & (M_{0,1})(M_{0,-1}),\ (M_{1,0})(M_{-1,0}) & (M_{1,0})(M_{0,-1}) \\ (M_{-1,0})(M_{-1,0}) & (M_{-1,0})(M_{0,-1}) & (M_{0,-1})(M_{0,-1}) \ . \end{array} \tag{7.26}$$

We emphasize that all these GIO's have vanishing mesonic charge, we are not counting the BPS operators such as $\det(A)$.

7.4 Refinement: Multicharges at Finite N

As with the refinement of the charges discussed in§3.2.1, it is simple to generalize the arguments of the previous subsection to partition functions g_1 depending on more than one variables, arising for instance from CY cones with isometry $U(1)^2$ or $U(1)^3$.

Consider a toric CY cone whose integer points are described by the set C. The single particle states are described by the generating function

$$g_1(t_1,t_2,t_3) = \sum_{n,m,r\in C} t_1^n t_2^m t_3^r \ . \tag{7.27}$$

Every point in C contributes once to g_1, i.e., we are considering a quantum particle whose states are precisely the integer points in the toric cone. The multi-trace generating function $g_N(t_1, t_2, t_3)$, in analogy with eq. (7.1), is given by

$$g(\nu, t_i) = \sum_N g_N(t_i)\nu^N = \prod_{n,m,r\in C} \frac{1}{(1-\nu t_1^n t_2^m t_3^r)} = \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k}{k} g_1(t_1^k, t_2^k, t_3^k)\right) . \quad (7.28)$$

The coefficients $g_N(t_i)$ can be interpreted as the multi-particle partition function of N boson whose single particle states are given by the integer points of C.

7.4.1 The Conifold Reloaded

Recalling eq. (4.4), the generating function g_1 is given by:

$$g_1 = \frac{xy(1-q^2)}{(1-qx)(1-qy)(q-x)(q-y)} = 1+q(x+y+\frac{1}{x}+\frac{1}{y})+q^2(1+\frac{1}{x^2}+x^2+\frac{1}{y^2}+y^2+xy+\frac{1}{xy}+\frac{x}{y}+\frac{y}{x})+\dots .$$

We can identify the charges (q, x, y) for following four operators

$$M_{0,1} = (1,1,0), \quad M_{0,-1} = (1,-1,0), \quad M_{0,1} = (1,0,1), \quad M_{0,-1} = (1,0,-1) .$$

Therefore, the generating function is, according to eq. (7.28),

$$\begin{aligned} g(\nu; q, x, y) \;=\; & \frac{1}{(1-\nu)} \frac{1}{(1-\nu qx)(1-\nu qy)(1-\nu\frac{q}{x})(1-\nu\frac{q}{y})} \frac{1}{(1-\nu q^2)(1-\nu q^2 x^2)(1-\nu\frac{q^2}{x^2})} \\ & \frac{1}{(1-\nu q^2 y^2)(1-\nu\frac{q^2}{y^2})(1-\nu q^2 xy)(1-\nu\frac{q^2}{xy})(1-\nu\frac{q^2 x}{y})(1-\nu\frac{q^2 y}{x})} . \end{aligned}$$

Now we try to apply above result. For $N = 1$ case we find the

$$g_1 = 1 + q\left(\frac{1}{x} + x + \frac{1}{y} + y\right) + q^2\left(1 + \frac{1}{x^2} + x^2 + \frac{1}{y^2} + \frac{1}{x\,y} + \frac{x}{y} + \frac{y}{x} + x\,y + y^2\right) + \dots ,$$

which has obvious correspondence with the variables $M_{i,j}$. For $N = 2$, we get the following expansion up to R-charge two

$$g_2 = 1+q\left(\frac{1}{x} + x + \frac{1}{y} + y\right)+q^2\left(3 + \frac{2}{x^2} + 2\,x^2 + \frac{2}{y^2} + \frac{2}{x\,y} + \frac{2\,x}{y} + \frac{2\,y}{x} + 2\,x\,y + 2\,y^2\right) .$$

Again, it is easy to find the mapping between the terms here and operators in eq. (7.25) and eq. (7.26).

7.5 Theories with only $U(1)^2$ symmetry

We can also start from a theory with only $U(1)^2$ symmetry, whose Poincaré series is given by

$$g_1(t_1, t_2) = \sum_{n,m\geq 0} a_{m,n} t_1^n t_2^m \ . \tag{7.29}$$

Using this we can find the generating function given by

$$g(\nu, t_i) = \sum_N g_N(t_i)\nu^N = \prod_{n,m\geq 0} \frac{1}{(1-\nu t_1^n t_2^m)^{a_{m,n}}} = \exp\left(\sum_{k=0}^{\infty} \frac{\nu^k}{k} g_1(t_1^k, t_2^k)\right) \ . \tag{7.30}$$

8 Conclusions and Prospects

In this paper we have considered the 1/2-BPS operators of generic superconformal quiver gauge theories, living on N D3-branes probing the tip of a Calabi-Yau (CY) cone. It is shown how to construct the explicit generating functions that count the scalar BPS operators. We have discussed in great detail various classes of CYs (orbifolds, toric varietes, del Pezzo's and complete intersections, even geometries for which the gauge theory is not yet known), providing a simple bridge (the "Plethystic Exponential") between the algebraic geometry of the CY and the generating functions of the BPS states.

The plethystics directly relate three different generating functions: (1) the defining equations of the CY (syzygies) as well as the moduli space of vacua, (2) the single-trace operators and (3) the multi-trace operators. Beautiful structures thus emerge, exhibiting a rich inter-play between quiver gauge theories, algebraic geometry, combinatorics and analytic number theory. This intricate framework allows us to solve the 3 problems posed in the introduction, whereby realising our wish-list.

There are a number of directions that could be pursued for future work. Let us discuss some of them. We only considered the subset of operators with vanishing baryonic charges. For instance, for the conifold we did not include, in the counting, the operator $\det(A)$. It would be nice to find the partition functions including the baryonic charge, that may be compared to analogous computations on the string, $AdS_5 \times X^5$, side.

A possible continuation of our work could be to extend the study of chiral 1/2-BPS operators in $\mathcal{N} = 1$ quivers to consider also 1/2-BPS operators with space-time angular momenta and 1/2-BPS fermionic operators. This would give partition functions depending on additional charges and would, for instance, enable a computation of the BPS index in quiver gauge theories, see [67, 68, 69, 70].

Another exension would be to consider 1/4-BPS operators, annihilated only by one supercharge. We remark that we are studying 1/2-BPS operators in $\mathcal{N} = 1$ gauge theories annihilated by 2 out of the 4 supercharges Q. These are the analogues of 1/8-BPS operators (annihilated by 2 out of the 16 supercharges) in $\mathcal{N} = 4$ SYM. It would be very interesting to extend the study to 1/4-BPS operators of quivers (annihilated by only 1 supercharge Q, analogous to 1/16-BPS ops in $\mathcal{N} = 4$ SYM) [12]. One possible outcome, for instance, could be a comparison with entropy counting of the recently constructed AdS_5 SUSY black holes [13].

In $\mathcal{N} = 4$ SYM an interesting problem is whether there is a change in the number of BPS operators changing the coupling. At zero coupling one expects more BPS states. Here, there is a precisely analogous question. It was shown in [28] that, in the moduli space of SCFTs corresponding to a given quiver gauge theory, there is a special point with enhanced chiral ring[14]. It was observed in [28] that at this special point, the growth of the number of single-trace, $N = \infty$, BPS mesons is exponential instead of that quadratic (as is the case on generic points of the moduli space of SCFTs). It would be interesting to study further this mechanism, that could lead at finite temperature to phase transitions.

On the gravity/string side of AdS/CFT, we should also find the same partition functions. For single-trace operators the result is well-known. The interesting case is multi-trace at finite N. One way to reproduce the g_N should be counting Giant Gravitons (GG's) in $AdS_5 \times X^5$. There are studies of GG's in $AdS_5 \times S^5$ and $AdS_5 \times T^{1,1}$ [71]. In the case of S^5, one considers the classical moduli space of GG's and is led to study N

[12]One single-trace example of such operators is given by $\mathrm{Tr}(O\,K)$, where O is a scalar BPS operator and K is the scalar SUSY partner of a conserved current: $\bar{Q}_\alpha O = 0$, and $Q^2 K = \bar{Q}^2 K = 0$, so $\mathrm{Tr}(O\,K)$ is annihilated only by $\bar{Q}^2$.

[13]Notice that these BPS black holes are constructed in 5-dimensional gauged supergravity with $U(1)^k$ gauge group, so can in principle be uplifted to various $AdS_5 \times X^5$ solutions.

[14]For $\mathcal{N} = 4$ and orbifolds thereof this special point is the free theory, for the conifold it corresponds to having vanishing superpotential. For generic quivers we have only one term of the superpotential vanishing.

classical non-interacting particles, whose single particle phase space is $\mathbb{C}^3$. Quantizing the multi particle phase space one gets N bosons whose single particle states are the integer points of the toric cone of $\mathbb{C}^3$. The partition function is precisely the finite N partition function of 1/8-BPS ops in $\mathcal{N}=4$ SYM. In generic toric quivers, for instance, one should find the result of the end of §7:

$$\sum_N g_N(t_i)\nu^N = \prod_{n,m,r\in C} \frac{1}{(1-\nu t_1^n t_2^m t_3^r)}$$

A different approach is [72], where they consider "dual GG's", i.e., D3-brane wrapping an S^3 inside AdS_5, and moving along a trajectory in X^5. For generic Sasaki-Einstein manifolds X^5, we conjecture that these states are BPS if and only if the trajectory is a BPS geodesic. We already know that single-trace BPS mesons are the quantization of such BPS geodesics (see [28]), so the final result should be a simple outcome of the use of the Plethystic Exponential on the gravity side.

Acknowledgements

We heartily acknowledge James Spraks, Dario Martelli and Davide Gaiotto for wonderful communications. S. B. is indebted to Antonello Scardicchio for discussions and the "Fondazione Angelo della Riccia" for financial support. B. F. is obliged to the Marie Curie Research Training Network under contract number MRTN-CT-2004-005104 and extends his thanks to Merton College, Oxford for warm reception at the final stages of the draft. A. H. is grateful to Alastair King, Marcos Mariño and Yuji Okawa for enlightening conversations, to Philip Candelas and Xenia de la Ossa for kind hospitality at the University of Oxford and Jerome Gauntlett, at Imperial College, London, as well as the Technion and Tel Aviv University as the manuscript was being completed. Y. H. H. bows to the gracious patronage, via the FitzJames Fellowship, of Merton College, Oxford.

References

[1] A. Hanany, talk at "QCD and String Theory," July 2-8, 2006, Ringberg Castle, Tegernsee, `http://www.mppmu.mpg.de/common/seminar/conf/conf2006/QCDST/qcdstprogram_dateien/ringberg_hanany.pdf`

[2] M. R. Douglas and G. W. Moore, "D-branes, Quivers, and ALE Instantons," arXiv:hep-th/9603167.

[3] C. V. Johnson and R. C. Myers, "Aspects of type IIB theory on ALE spaces," Phys. Rev. D **55**, 6382 (1997) [arXiv:hep-th/9610140].

[4] S. Kachru and E. Silverstein, "4d conformal theories and strings on orbifolds," Phys. Rev. Lett. **80**, 4855 (1998) [arXiv:hep-th/9802183].

[5] A. E. Lawrence, N. Nekrasov and C. Vafa, "On conformal field theories in four dimensions," Nucl. Phys. B **533**, 199 (1998) [arXiv:hep-th/9803015].

[6] A. Hanany and Y. H. He, "Non-Abelian finite gauge theories," JHEP **9902**, 013 (1999) [arXiv:hep-th/9811183].

[7] A. Hanany and Y. H. He, "A monograph on the classification of the discrete subgroups of SU(4)," JHEP **0102**, 027 (2001) [arXiv:hep-th/9905212].

[8] M. R. Douglas, B. R. Greene and D. R. Morrison, "Orbifold resolution by D-branes," Nucl. Phys. B **506**, 84 (1997) [arXiv:hep-th/9704151].

[9] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," Adv. Theor. Math. Phys. **3**, 1 (1999) [arXiv:hep-th/9810201].

[10] C. Beasley, B. R. Greene, C. I. Lazaroiu and M. R. Plesser, "D3-branes on partial resolutions of abelian quotient singularities of Calabi-Yau threefolds," Nucl. Phys. B **566**, 599 (2000) [arXiv:hep-th/9907186].

[11] B. Feng, A. Hanany and Y. H. He, "D-brane gauge theories from toric singularities and toric duality," Nucl. Phys. B **595**, 165 (2001) [arXiv:hep-th/0003085].
B. Feng, A. Hanany and Y. H. He, "Phase structure of D-brane gauge theories and toric duality," JHEP **0108**, 040 (2001) [arXiv:hep-th/0104259].

B. Feng, S. Franco, A. Hanany and Y. H. He, "Symmetries of toric duality," JHEP **0212**, 076 (2002) [arXiv:hep-th/0205144].

[12] B. Feng, A. Hanany, Y. H. He and A. M. Uranga, "Toric duality as Seiberg duality and brane diamonds," JHEP **0112**, 035 (2001) [arXiv:hep-th/0109063].

[13] C. E. Beasley and M. R. Plesser, "Toric duality is Seiberg duality," JHEP **0112**, 001 (2001) [arXiv:hep-th/0109053].

[14] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, "A geometric unification of dualities," Nucl. Phys. B **628**, 3 (2002) [arXiv:hep-th/0110028].

[15] B. Feng, A. Hanany, Y. H. He and A. Iqbal, "Quiver theories, soliton spectra and Picard-Lefschetz transformations," JHEP **0302**, 056 (2003), hep-th/0206152.
A. Hanany and J. Walcher, "On duality walls in string theory," JHEP **0306**, 055 (2003) [arXiv:hep-th/0301231].
S. Franco, Y. H. He, C. Herzog and J. Walcher, "Chaotic cascades for D-branes on singularities," arXiv:hep-th/0412207.
C. P. Herzog, "Seiberg duality is an exceptional mutation," JHEP **0408**, 064 (2004) [arXiv:hep-th/0405118].

[16] A. Hanany and K. D. Kennaway, "Dimer models and toric diagrams," arXiv:hep-th/0503149.

[17] S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, "Brane dimers and quiver gauge theories," JHEP **0601**, 096 (2006) [arXiv:hep-th/0504110].

[18] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, "Gauge theories from toric geometry and brane tilings," JHEP **0601**, 128 (2006) [arXiv:hep-th/0505211].

[19] A. Butti, D. Forcella and A. Zaffaroni, "The dual superconformal theory for L(p,q,r) manifolds," JHEP **0509**, 018 (2005) [arXiv:hep-th/0505220].

[20] A. Butti and A. Zaffaroni, "R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization," JHEP **0511**, 019 (2005), hep-th/0506232.
– "From toric geometry to quiver gauge theory: The equivalence of a-maximization and Z-minimization," Fortsch. Phys. **54**, 309 (2006) [arXiv:hep-th/0512240].

[21] A. Hanany and D. Vegh, "Quivers, tilings, branes and rhombi," arXiv:hep-th/0511063.

[22] B. Feng, Y. H. He, K. D. Kennaway and C. Vafa, "Dimer models from mirror symmetry and quivering amoebae," arXiv:hep-th/0511287.

[23] S. Franco and D. Vegh, "Moduli spaces of gauge theories from dimer models: Proof of the correspondence," arXiv:hep-th/0601063.

[24] A. Hanany, C. P. Herzog and D. Vegh, "Brane tilings and exceptional collections," arXiv:hep-th/0602041.

[25] I. Garcia-Etxebarria, F. Saad and A. M. Uranga, "Quiver gauge theories at resolved and deformed singularities using dimers," JHEP **0606**, 055 (2006) [arXiv:hep-th/0603108].

[26] Y. Imamura, "Anomaly cancellations in brane tilings," arXiv:hep-th/0605097.

[27] A. Butti, D. Forcella and A. Zaffaroni, "Deformations of conformal theories and non-toric quiver gauge theories," arXiv:hep-th/0607147.

[28] S. Benvenuti and M. Kruczenski, "Semiclassical strings in Sasaki-Einstein manifolds and long operators in N = 1 gauge theories," arXiv:hep-th/0505046.
S. Benvenuti and M. Kruczenski, "From Sasaki-Einstein spaces to quivers via BPS geodesics: L(p,q—r)," JHEP **0604**, 033 (2006) [arXiv:hep-th/0505206].

[29] K. Ueda and M. Yamazaki, "Brane tilings for parallelograms with application to homological mirror symmetry," arXiv:math.ag/0606548.

[30] K. Ueda and M. Yamazaki, "A Note on Brane Tilings and McKay Quivers," arXiv:math.ag/0605780.

[31] A. Hanany and A. Iqbal, "Quiver theories from D6-branes via mirror symmetry," JHEP **0204**, 009 (2002) [arXiv:hep-th/0108137].
M. Wijnholt, "Large volume perspective on branes at singularities," Adv. Theor. Math. Phys. **7**, 1117 (2004) [arXiv:hep-th/0212021].
S. Franco, A. Hanany and P. Kazakopoulos, "Hidden exceptional global symmetries in 4d CFTs," JHEP **0407**, 060 (2004) [arXiv:hep-th/0404065].

[32] D. Martelli, J. Sparks and S. T. Yau, "The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds," arXiv:hep-th/0503183.

[33] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," Nucl. Phys. B **536**, 199 (1998) [arXiv:hep-th/9807080].

[34] J. Park, R. Rabadan and A. M. Uranga, "Orientifolding the conifold," Nucl. Phys. B **570**, 38 (2000) [arXiv:hep-th/9907086].

[35] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[36] N. Nekrasov and S. Shadchin, "ABCD of instantons," Commun. Math. Phys. **252**, 359 (2004), [arXiv:hep-th/0404225].

[37] D. Martelli, J. Sparks and S. T. Yau, "Sasaki-Einstein manifolds and volume minimisation," arXiv:hep-th/0603021.

[38] S.-T. Yau and Y. Yu, "Gorenstein Quotients Singularities in Dimension Three," Memoirs of the AMS, 505, 1993.

[39] H. F. Blichfeldt, "Finite Collineation Groups". Univ. Chicago Press, Chicago, 1917.

[40] P. Slodowy, "Simple singularities and simple algebraic groups," in Lecture Notes in Mathematics 815, Springer, Berlin, 1980.
John McKay, "Graphs, singularities and finite groups," Proc. Symp. Pure Math. vol. 37, Amer. Math. Soc., Providence, Rhode Island, 1980, pp. 183-186.

[41] B. A. Burrington, J. T. Liu and L. A. Pando Zayas, "Finite Heisenberg groups in quiver gauge theories," Nucl. Phys. B **747**, 436 (2006) [arXiv:hep-th/0602094].

[42] D. Grayson and M. Stillman, "Macaulay 2, a software system for research in algebraic geometry." Available at `http://www.math.uiuc.edu/Macaulay2/`

[43] J. Gray, Y. H. He, V. Jejjala and B. D. Nelson, "Exploring the vacuum geometry of N = 1 gauge theories," arXiv:hep-th/0604208. Nucl.Phys. B. (2006).

[44] J. Gray, Y. H. He and A. Lukas, "Algorithmic algebraic geometry and flux vacua," arXiv:hep-th/0606122.

[45] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2006, (`http://www.gap-system.org`).

[46] O. Aharony, A. Hanany and B. Kol, "Webs of (p,q) 5-branes, five dimensional field theories and grid diagrams," JHEP **9801**, 002 (1998) [arXiv:hep-th/9710116].

[47] M. Brion, "Points entiers dans les polydres convexes," Annales Scientifiques de l'École Normale Supérieure (4) tome 21, numro 4 (1988), p. 653-663.
M. Beck, C. Haase and F. Sottile, "Theorems of Brion, Lawrence, and Varchenko on rational generating functions for cones," math.CO/0506466.
T. Huettemann, "On a theorem of Brion," math.CO/0607297.

[48] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," Commun. Math. Phys. **262**, 51 (2006) [arXiv:hep-th/0411238].

[49] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, "An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals," JHEP **0506**, 064 (2005) [arXiv:hep-th/0411264].

[50] S. Benvenuti, A. Hanany and P. Kazakopoulos, "The toric phases of the Y(p,q) quivers," JHEP **0507**, 021 (2005) [arXiv:hep-th/0412279].
S. Benvenuti and A. Hanany, "Conformal manifolds for the conifold and other toric field theories," JHEP **0508**, 024 (2005) [arXiv:hep-th/0502043].
A. Hanany, P. Kazakopoulos and B. Wecht, "A new infinite class of quiver gauge theories," JHEP **0508**, 054 (2005) [arXiv:hep-th/0503177].
H. Kihara, M. Sakaguchi and Y. Yasui, "Scalar Laplacian on Sasaki-Einstein manifolds Y(p,q)," Phys. Lett. B **621**, 288 (2005) [arXiv:hep-th/0505259].

[51] B. Feng, S. Franco, A. Hanany and Y. H. He, "Unhiggsing the del Pezzo," JHEP **0308**, 058 (2003) [arXiv:hep-th/0209228].

[52] M. Aganagic, A. Karch, D. Lust and A. Miemiec, "Mirror symmetries for brane configurations and branes at singularities," Nucl. Phys. B **569**, 277 (2000) [arXiv:hep-th/9903093].

[53] J. M. F. Labastida and M. Marino, "A new point of view in the theory of knot and link invariants," arXiv:math.qa/0104180.

[54] E. Getzler, M.M. Kapranov, "Modular operads,' dg-ga/9408003.

[55] W. Lerche, P. Mayr and N. P. Warner, "Non-critical strings, del Pezzo singularities and Seiberg-Witten curves," Nucl. Phys. B **499**, 125 (1997) [arXiv:hep-th/9612085].

[56] H. Verlinde and M. Wijnholt, "Building the standard model on a D3-brane," arXiv:hep-th/0508089.

[57] D. Berenstein, V. Jejjala and R. G. Leigh, "The standard model on a D-brane," Phys. Rev. Lett. **88**, 071602 (2002) [arXiv:hep-ph/0105042].

[58] R. Hartshorne, "Algebraic Geometry," volume 52, Graduate Texts in Mathematics. Springer-Verlag.

[59] B. Harbourne, "Problems and progress: survey on fat points in P2", math.AG/0101112

[60] J. P. Gauntlett, D. Martelli, J. Sparks and S. T. Yau, "Obstructions to the existence of Sasaki-Einstein metrics," arXiv:hep-th/0607080.

[61] D. Conti, "Cohomogeneity one Einstein-Sasaki 5-manifolds," math.DG/0606323.

[62] G. Meinardus, "Asymptotische Aussagen über Partitionen," Mathematische Zeitschrift 59 (1954) 388–398.

[63] A. A. Actor, "Infinite products, partition functions, and the Meinardus theorem," J. Mathematical Physics, Vol 35, 11, Nov. 1994, pp.5749-5764.

[64] M. Green, J. Schwarz and E. Witten, " Superstring theory," Cambridge University Press (1987).

[65] H. Baker, "Percy Alexander MacMahon," J. Lond. Math. Soc. 5 (1930), 305-318. A. Okounkov, N. Reshetikhin and C. Vafa, "Quantum Calabi-Yau and classical crystals," arXiv:hep-th/0309208.

[66] N. Halmagyi, A. Sinkovics and P. Sulkowski, "Knot invariants and Calabi-Yau crystals," JHEP **0601**, 040 (2006) [arXiv:hep-th/0506230].

[67] C. Romelsberger, "An index to count chiral primaries in N = 1 d = 4 superconformal field theories," arXiv:hep-th/0510060.

[68] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, "An index for 4 dimensional super conformal theories," arXiv:hep-th/0510251.

[69] Y. Nakayama, "Index for orbifold quiver gauge theories," Phys. Lett. B **636**, 132 (2006) [arXiv:hep-th/0512280].
Y. Nakayama, "Index for supergravity on AdS(5) x T**(1,1) and conifold gauge theory," arXiv:hep-th/0602284.
A. Ceresole, G. Dall'Agata, R. D'Auria and S. Ferrara, "Spectrum of type IIB supergravity on AdS(5) x T(11): Predictions on N = 1 SCFT's," Phys. Rev. D **61**, 066001 (2000) [arXiv:hep-th/9905226].
S. S. Gubser, "Einstein manifolds and conformal field theories," Phys. Rev. D **59**, 025006 (1999) [arXiv:hep-th/9807164].

[70] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, "Supersymmetric states of N = 4 Yang-Mills from giant gravitons," arXiv:hep-th/0606087.

[71] A. Mikhailov, "Giant gravitons from holomorphic surfaces," JHEP **0011**, 027 (2000) [arXiv:hep-th/0010206].
D. Berenstein, C. P. Herzog and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," JHEP **0206**, 047 (2002) [arXiv:hep-th/0202150].
C. E. Beasley, "BPS branes from baryons," JHEP **0211**, 015 (2002), 0207125.

[72] G. Mandal and N. V. Suryanarayana, "Counting 1/8-BPS dual-giants," arXiv:hep-th/0606088.