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Research Article

Chern-Simons: Fano and Calabi-Yau

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We present the complete classification of smooth toric Fano threefolds, known to the algebraic geometry literature, and perform some preliminary analyses in the context of brane tilings and Chern-Simons theory on M2-branes probing Calabi-Yau fourfold singularities. We emphasise that these 18 spaces should be as intensely studied as their well-known counterparts: the del Pezzo surfaces.

1. Introduction

A flurry of activity has, since the initial work of Bagger and Lambert [1–3] and Gustavsson [4], rather excited the community for the past two years upon the subject of supersymmetric Chern-Simons theories. It is by now widely believed that the world-volume theory of M2-branes on various backgrounds is given by a $(2+1)$-dimensional quiver Chern-Simons (QCS) theory [5–26], most conveniently described by a brane tiling.

Even though analogies with the case of D3-branes in Type IIB, whose world-volume theory is a $(3 + 1)$-dimensional supersymmetric quiver gauge theory, are very reassuring, the story is much less understood for the M2 case. Much work has been devoted to the understanding of issues such as orbifolding, phases of duality, brane tilings, and dimer/crystal models and so forth. Nevertheless, the role played by the correspondence between the world-volume theory and the underlying Calabi-Yau geometry is of indubitable importance. Indeed, there is a bijection: the vacuum moduli space of the former is, tautologically, the latter, while the geometrical engineering on the latter gives, by construction, the former. This bijection, called, respectively, the "forward" and "inverse" algorithms [27, 28], persists in any dimension and can be succinctly summarised in Table 1.

Table 1: Brane probes and associated world-volume physics in various backgrounds.

Brane probe	Theory	Background	World-volume theory	Vacuum moduli space
D5	Type IIB	$\mathbb{R}^{1,5} \times CY2$	(5+1)-d $\mathcal{N} = 1$ gauge theory	CY2
D3	Type IIB	$\mathbb{R}^{1,3} \times CY3$	(3+1)-d $\mathcal{N} = 1$ gauge theory	CY3
M2	M-theory	$\mathbb{R}^{1,2} \times CY4$	(2+1)-d $\mathcal{N} = 2$ Chern-Simons	CY4

A crucial feature for all the brane embeddings in Table 1 is that in the toric case they are all described by brane tilings. The first case, with CY2, is described by one-dimensional tilings, that is, brane intervals and thus brane constructions following the work in [29]. The second case is the well-established two-dimensional brane tilings which use dimer techniques to study supersymmetric gauge theories [30–32]. The third case is the newly proposed construction [13] of Chern-Simon theories.

It is perhaps naïvely natural to propose three-dimensional tilings for the case of M2-branes probing CY4, but in fact, it turns out not to be as useful as it may initially seem. These three-dimensional tilings have been nicely advocated in the crystal model [33, 34]. The main issue perhaps is the current shortcoming of this model to identify the gauge groups with a simplex as it is done for the tilings in dimensions one and two. In the one-dimensional case for toric CY2, the gauge group is identified with an edge of the tiling, and the matter content with nodes. For the two-dimensional case for toric CY3, the gauge fields, matter fields, and interactions are, respectively, identified with faces, edges, and nodes of the tiling. But for the proposed crystal model, there is no such simple interpretation yet known.

We are thus led, for now, to keep on the path of two-dimensional tilings, while bearing in mind that the data needed to specify a QCS theory is given by gauge groups, matter fields, and interactions, as well as the additional data of the CS levels for the gauge groups. These nicely map, respectively, to tiles, edges, and nodes, while the corresponding CS levels are given by fluxes on the tiles. It would be interesting to check if this correspondence between tilings in one and two dimensions, that is, for toric Calabi-Yau n-folds with $n = 2, 3, 4$, can be extended to possibly higher-dimensional tilings and perhaps higher-dimensional Calabi-Yau spaces.

The cases for Calabi-Yau two- and threefolds are well established over the past decade. These are affine complex cones over base complex curves and surfaces, or real cones over real, compact, Sasaki-Einstein three and five manifolds. Perhaps the most extensively studied are, inspired by phenomenological concerns, D3-branes and Calabi-Yau threefolds and the widest class studied therein is *toric Calabi-Yau cones*. A rather complete picture for both the forward and inverse algorithms, as well as the unifying perspective of brane tilings and dimer models, has emerged over the last decade. Ricci-flat metrics have even been found for infinite families within the class of these noncompact spaces.

Another crucial family of Calabi-Yau threefold cones affords a clear construction, and the world-volume physics has been intensely investigated (cf., e.g., [35–37]). The base surfaces here are so-called *del Pezzo* surfaces which afford positive curvature, so that the appropriate cones over them have just the right behaviour to make the affine threefold have zero Ricci curvature. More precisely, these surfaces are dP_n, which is $\mathbb{P}^2$ blowup at n equal to zero up to eight generic points, or the zeroth Hirzebruch surface $F_0 := \mathbb{P}^1 \times \mathbb{P}^1$. In fact, the cones over F_0 and $dP_{n=0,1,2,3}$ are toric, whereby making these five del Pezzo members of particular interest. The $(3 + 1)$-dimensional gauge theories for these were first constructed in [27, 28], giving rise to such interesting phenomena as toric duality and tilings.

Indeed, all toric gauge theories in (3 + 1) dimensions obey a remarkable topological formula: take the number of nodes in the quiver, the number of fields, and the number of terms in the superpotential; their alternating sum vanishes. This is a key for the powerful brane tiling (dimer model) description (cf. review in [32]) of these theories. Interestingly, this relation is still obeyed for the myriad of all known (2 + 1)-dimensional QCS theories to date and suggests that a planar brane tiling may still be the underlying principle behind theories living on M2-branes probing affine Calabi-Yau fourfolds. The richness of the (3 + 1)-dimensional theories beckons for their analogous and extensions to the (2 + 1)-dimensional case.

It is therefore a natural and important question to ask what are the corresponding geometries for Calabi-Yau fourfolds and physically what are the associated (2 + 1)-dimensional QCS theories on the M2-brane world volume, that is, what are the (smooth) toric complex threefolds which admit positive curvature? Based on the ample experience with and the wealth of physics engendered by the aforementioned five del Pezzo cases for threefolds, these fourfolds could hold a key toward understanding QCS and M2 theories.

It is the purpose of the current short note, a prologue to [38], to present the *dramatis personae* onto the stage and to introduce some rudiments of their properties as well as to initiate the first constructions of the QCS physics associated thereto. Indeed, complex manifolds admitting positive curvature are in general called *Fano varieties* of which the del Pezzo surfaces are merely the two-dimensional examples. We will see that a complete and convenient classification exists for the smooth toric Fano threefolds over which Calabi-Yau four-fold cones can be established; we will take advantage of the existing data and use the forward algorithm to explicitly construct the quivers, superpotentials, and Chern-Simons levels for some cases. A companion paper, of substantially more length and in-depth analysis [38], will ensue in the near future. It is our hope that the 18 characters to which we draw your attention will, in due course, become as familiar as the del Pezzo family to the community.

2. Fano Varieties

Fano varieties are of obvious importance; these are varieties which admit an ample anticanonical sheaf; thus, whereas Calabi-Yau varieties are of zero curvature, they are of positive curvature. (Recently, lower bounds on the Ricci curvature of Fano manifolds have been found [39].) Therefore, not only could Fano varieties constitute cycles of positive volume that can shrink inside a Calabi-Yau, but also, could they provide local models of Calabi-Yau of a higher dimension. This second case is perhaps of more interest in the brane-probe scenario where the transverse directions to the branes are affine, noncompact Calabi-Yau spaces. In particular, one could construct an affine complex cone over a Fano n-fold, so as to construct a Calabi-Yau $(n+1)$-fold, and the branes then reside at the tip of the cone. This situation has become well known to the AdS/CFT correspondence.

What are explicit examples of Fano varieties? In complex dimension one, there is only $\mathbb{P}^1$, the sphere, which obviously has positive curvature. In dimension two, they are called del Pezzo surfaces. In particular, they are $\mathbb{P}^2$, as well its blowup dP_n at $n = 1$ up to $n = 8$ generic points thereon, and the zeroth Hirzebruch surface $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$. Of these 10, $\mathbb{P}^2$, $\mathbb{F}_0$, and dP_n for $n = 1, 2, 3$ admit a toric description. These have been used extensively in constructing gauge theories on the D3-brane world volume [27–40], and the moduli spaces of these theories are correspondingly local Calabi-Yau threefolds.

We point out that, of course, the aforementioned are *smooth* Fano varieties. Indeed, we can readily construct affine Calabi-Yau spaces which are singular cones. For example, for complex dimension one, we indeed have the smooth $\mathbb{P}^1$, leading to the affine Calabi-Yau 2-fold $\mathbb{C}^2/\mathbb{Z}_2$, with the corresponding quiver gauge theory in $(5+1)$ dimensions, but we also have any of the famous ADE singularities given by $\mathbb{C}^2$ quotient by a discrete subgroup of $SU(2)$ which give rise to well-known gauge theories. In complex dimension 2, we have $\mathbb{P}^2$, corresponding to the affine Calabi-Yau 3-fold $dP_0 = \mathbb{C}^3/\mathbb{Z}_3$; however, any $\mathbb{C}^3/\mathbb{Z}_n$ is just as good with a singular base Fano 2-fold in a weighted projected space.

Our chief interest lies in the situation of dimension three. These Fano threefolds can give rise to Calabi-Yau fourfolds which can then be probed by M2-branes in order to arrive at quiver Chern-Simons (QCS) theories on their world volume. A classification of the Fano vareities was achieved in the 80s [41–43]; there is a wealth thereof. Our particular interest will once more be on the toric Fano threefolds where such techniques as tilings and dimers will be conducive. Toric Fano threefolds have been studied in [44, 45]. In dimension n, an obvious general class of toric Fano k-folds is $\prod_j \mathbb{P}^{k_j}$ where $\{k_j\}$ is a partition of n, that is, $n = \sum_j k_j$.

With the rapid advance of computer algebra and algorithmic algebraic geometry, especially in applications to physics (cf. [46–48]), even non-smooth Fano varieties can be classified [50]. (Indeed, in any dimension d, it is known that there are a finite number of *smooth* Fano varieties [49].) A convenient database has been established whereby one could readily search within an online depository [51]. (We are grateful to Richard Thomas for revealing this treasure trove to us.)

2.1. Smooth Toric Fano Threefolds

Given the enormity of the number, we were to allow singularities—against which, physically, there need be no prejudice—and being inspired by the 2-fold case of the del Pezzo surfaces all being smooth, we will henceforth restrict our attention to the *smooth* toric Fano threefolds. In the parlance of toric geometry, the corresponding cone is called regular. There is a total of 18 such threefolds, a reasonable set indeed. We will adhere to the standard notation of [45] wherein the family is tabulated and also to the identifier with the database [51] for the sake of canonical reference. This is presented in Table 2.

2.1.1. Toric Data

Some detailed explanation of the nomenclature in Table 2 is in order. The toric data is such that the columns are vectors which generate the cone of the variety; in the D-brane context, this has become known as the G_t matrix. Note that each is a 3-vector, signifying that we are dealing with threefolds. Moreover, the point $(0,0,0)$ is always an internal point. This property is equivalent to the Fano condition. Indeed, as we recall from [27, 28], the del Pezzo surfaces all have a single internal point. The explicit topology of each space is also given, following [45].

Indeed, our interest in (compact) Fano threefolds is that the complex cone thereupon is an (noncompact) affine Calabi-Yau fourfold which M2-branes may probe. Going form the data in the table to the fourfold is simple; we only need to add one more dimension, say, a row of 1s to each of the matrices. In such cases, the geometry will be cones over what is reported in the third column. In the physics literature, there have been several cases which

Table 2: The 18 smooth toric Fano threefolds. For full explanation of notation, see the second paragraph of Section 2.1 and those that follow.

	Id of [51]	G_t: toric data	Geometry	$(b_2,\ g,\ Sym)$
$\mathbb{P}^3$	4	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^3$	$(1, 33, U(4))$
$\mathcal{B}_1$	35	$\begin{pmatrix} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	$(2, 32, [3, 1^2])$
$\mathcal{B}_2$	36	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	$(2, 29, [3, 1^2])$
$\mathcal{B}_3$	37	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$	$(2, 28, [2^2, 1^2])$
$\mathcal{B}_4$	24	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^2 \times \mathbb{P}^1$	$(2, 28, [3, 2, 1])$
$\mathcal{C}_1$	105	$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$	$(3, 27, [2^2, 1^2])$
$\mathcal{C}_2$	136	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{dP_1} \oplus \mathcal{O}_{dP_1}(\ell)), \qquad \ell^2\vert_{dP_1} = 1$	$(3, 26, [2, 1^3])$
$\mathcal{C}_3$	62	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(3, 25, [2^3, 1])$
$\mathcal{C}_4$	123	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$	$dP_1 \times \mathbb{P}^1$	$(3, 25, [2^2, 1^2])$
$\mathcal{C}_5$	68	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1))$	$(3, 23, [2^2, 1^2])$
$\mathcal{D}_1$	131	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{P}^1$-blowup of $\mathcal{B}_2$	$(3, 26, [2, 1^3])$
$\mathcal{D}_2$	139	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^1$-blowup of $\mathcal{B}_4$	$(3, 24, [2, 1^3])$
$\mathcal{E}_1$	218	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$	dP_2 bundle over $\mathbb{P}^1$	$(4, 24, [2, 1^3])$
$\mathcal{E}_2$	275	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$	dP_2 bundle over $\mathbb{P}^1$	$(4, 23, [2, 1^3])$
$\mathcal{E}_3$	266	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$	$dP_2 \times \mathbb{P}^1$	$(4, 22, [2, 1^3])$
$\mathcal{E}_4$	271	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$	dP_2 bundle over $\mathbb{P}^1$	$(4, 21, [2, 1^3])$
$\mathcal{F}_1$	324	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$dP_3 \times \mathbb{P}^1$	$(5, 19, [2, 1^3])$
$\mathcal{F}_2$	369	$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	dP_3 bundle over $\mathbb{P}^1$	$(5, 19, [2, 1^3])$

have been studied in considerable depth and detail: the cone over $\mathbb{P}^3$ is the orbifold $\mathbb{C}^4/\mathbb{Z}_4$, the Sasaki-Einstein 7-fold (a homogeneous space which is a circle fibration over the $\mathbb{P}^1 \times \mathbb{P}^2$), which is a *real* cone over $\mathcal{B}_4$, is dubbed $M^{1,1,1}$ (see [13]), and the real Sasaki-Einstein cone over $\mathcal{C}_3$ is called $Q^{1,1,1}/\mathbb{Z}_2$ (cf., e.g., [19, 24, 25]).

2.1.2. Fibrations and Bundles

We, of course, recognise $\mathbb{P}^3$ (succeeding the sequence of $\mathbb{P}^1$ in dimension 1 and $\mathbb{P}^2 = dP_0$ in dimension 2) and the natural generalisation $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of $\mathbb{F}_0$. Indeed, in k complex dimensions, $\mathbb{P}^k$ and $(\mathbb{P}^1)^{\times k}$ are always smooth, toric, and Fano. The toric del Pezzo surfaces $dP_{0,1,2,3}$ also appear in Table 2, either in direct product or as various fibers. The notation $\mathbb{P}()$ means projectivisation so as to manufacture a compact project threefold. Indeed, we are primarily interested in the *affine* Calabi-Yau four-fold cone over these Fano threefolds, so the spaces in which we have interest do not need this projectivisation; we have included them for consistency of notation in that we are discussing the Fano threefolds in this section.

Therefore, the cone in a sense undoes the said projectivisation, and the fourfold is simply the total space of the fibration. For example, $\mathcal{B}_1$ is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$; here, $\mathcal{O}_{\mathbb{P}^2}(d)$ is a line bundle of degree d over $\mathbb{P}^2$, hence the fiber of $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ is of dimension $1 + 1 = 2$, which together with the base $\mathbb{P}^2$ dictates the total space as being of dimension $2 + 2 = 4$. (Of course, in line with standard notation $\mathcal{O}$ is the structure sheaf, or the line bundle of degree 0.) Subsequently, the projectivisation is of dimension $4 - 1 = 3$, as required. The actual affine Calabi-Yau fourfold is simply the total space $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$.

2.1.3. Symmetries

One piece of information, obviously of great importance, is the symmetry of the variety, which is encoded in the world-volume physics, either manifestly or as hidden global symmetries [52–55]. Inspecting the toric diagrams, we readily see that our list of Fano threefolds affords the following symmetries. The most symmetric case is, of course, $\mathbb{P}^3$, the cone over it has a full $U(4)$, acting as unitary transformations on the four coordinates. Next, both $\mathcal{B}_1$ and $\mathcal{B}_2$ have $SU(3) \times U(1)^2$, with $SU(3)$ acting on the base $\mathbb{P}^2$ and $U(1)$ for each fiber. Similarly, $\mathcal{B}_3$ has symmetry $SU(2)^2 \times U(1)^2$, with $SU(2)$ for the base $\mathbb{P}^1$, another $U(2)$ for the 2 identical line bundles $\mathcal{O}_{\mathbb{P}^1}$, and one more $U(1)$ for $\mathcal{O}(1)_{\mathbb{P}^1}$. Likewise, $\mathcal{B}_4$ has $SU(3) \times SU(2) \times U(1)$, with the $SU(3)$ and $SU(2)$ for the $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively, and $U(1)$ for the cone which gives the affine Calabi-Yau 4-fold. Proceeding along the same vein, $\mathcal{C}_1$, $\mathcal{C}_4$, and $\mathcal{C}_5$ share the symmetry $SU(2)^2 \times U(1)^2$, $\mathcal{C}_2$ has $SU(2) \times U(1)^3$, and $\mathcal{C}_3$ has $SU(2)^3 \times U(1)$. All remaining cases, namely, the $\mathfrak{D}$'s, $\mathcal{E}$'s, and $\mathfrak{F}$'s, are of symmetry $SU(2) \times U(1)^3$.

Note that the rank of the group of symmetries must total to 4 because we are dealing with a toric (affine) Calabi-Yau 4-fold. Indeed, one $U(1)$ factor of the symmetry is the R-symmetry and the remaining rank 3 symmetry, a global mesonic symmetry (cf. [52, 53]), and there could be possible additional $U(1)$-baryonic symmetries. We have summarised these mesonic symmetries in the last column of Table 2, under the entry *Sym*. Unless explicitly written, we have used the short-hand notation that

$$\left[3^{k_3}, 2^{k_2}, 1^{k_1}\right] := SU(3)^{k_3} \times SU(2)^{k_2} \times U(1)^{k_1}. \tag{2.1}$$

We note that the three cases of there being only a *single* $U(1)$ symmetry, namely, $\mathbb{P}^3$ (for which $U(4)$ contains the $U(1)$), $\mathcal{B}_4$, and $\mathcal{C}_3$, are products of projective spaces corresponding to the three partitions of 3. The corresponding QCS theories for these have been already constructed in the literature. This is perhaps unsurprising given the high degree of symmetry for these spaces.

2.1.4. Some Geometrical Data

We have also listed, to the rightmost of the table, some geometrical data, such as topological invariants. In particular, we tabulate the second Betti number b_2 and the genus g. Indeed, $b_2 = E - 3$, where E is the number of external points in the toric diagram, or since there is always a single internal point as discussed above, E is the number of columns of G_t minus 1. Now, recall that in the D3-brane probes on Calabi-Yau threefold case, the external vertices count the conserved anomaly-free global charges of the $(3 + 1)$-dimensional gauge theory. Each external vertex in the toric diagram is a divisor, and its corresponding charge gives rise to a basis for the set of mesonic, and baryonic charges: one of which is the R-symmetry, three of which are mesonic and the remaining $E - 4$ charges are baryonic.

However, in our present case of M2-branes probing the Calabi-Yau fourfold, the world-volume Chern-Simons theory in $(2+1)$ dimensions has no notion of anomaly, and hence there is no distinction between anomalous and anomaly-free baryonic charges. (An exception to this is the parity anomaly where one starts with a theory that has no CS terms, and one-loop perturbation theory generates a nonzero CS term. Since the CS term is odd under parity, one says that parity is conserved in the classical level but broken by a one-loop effect, hence anomalous. This is the only instance in which one can have anomalies in $(2 + 1)$ dimensions. Nevertheless, all the theories we deal with are protected by supersymmetry and, as long as the ranks are equal, the CS levels do not get quantum corrections (cf. [56]).) Thus, b_2 seems to be counting the number of baryonic charges if we extend the analogy from the $(3 + 1)$-dimensional situation.

On the other hand, a conserved baryonic charge corresponds to a gauge field in AdS. This is counted by the number of 2-cycles in the Sasaki-Einstein 7-fold (SE7), given by the 3-form on each 2-cycle. The number of 2-cycles in the SE7 is equal to the number of 5-cycles by Poincaré duality, which is in turn equal to the number E of external points in the toric diagram subtracted by 4. That is, the baryonic symmetries also afford a nice geometrical interpretation here: the number of columns of G_t is $E + 1$, then the number of baryonic symmetries is $E - 4$, signifying $U(1)^{E-4}$ (cf. of [25, Section 2] and also [23]). Then, since the second Betti number is $E - 3$, we have the number of baryonic symmetries as the topological quantity $b_2 - 1$.

Next, let us discuss the genus g. Note that a polarisation can be chosen as the ample anticanonical sheaf $A = K_X^{-1}$, which, due to its ampleness, can be used to embed into a projective space. It turns out that this embedding is of degree $d = c_1(X)^3$ into $\mathbb{P}^{g+1}$ such that $d = 2g - 2$. Of physical importance is that the $g + 2$ homogeneous coordinates of the ambient $\mathbb{P}^{g+1}$ constitute $g + 2$ gauge invariant chiral operators which parameterise the supersymmetric vacuum moduli space, with the relations satisfied amongst them providing the explicit equation thereof. In short, the number of generators of the moduli space is $g + 2$.

2.1.5. Hilbert Series

Now, it was first pointed out in [57, 58] that the Hilbert series of an algebraic variety is central toward understanding the gauge invariant operators of the gauge theory living on the branes probing the variety. For our purposes, this is a rational function which is the generating function for counting the spectrum of operators; it could be multivariate, having a number of "chemical potentials," which we call the refined Hilbert series, or it could depend on a single grading, which we call the unrefined Hilbert series. In particular, cones over the Fano twofolds, that is, the del Pezzo surfaces, have an elegant expression for their unrefined Hilbert series. We recall, [57, Section 3.3.1], that for the nth del Pezzo, of degree $9 - n$, it is $f(t; dP_n) = 1 + ((7-n)t + t^2/(1-t)^3)(n = 0, \ldots, 8)$; Note that $\mathbb{F}_0$ has the same unrefined Hilbert series as that of dP_1 though the refined, multi-variate Hilbert series does differentiate the two.

The unrefined Hilbert series, computed for the canonical embedding stated above, is also presented in [51], though perhaps not of immediate use since they are given as series expansions. We have recomputed these as rational functions. By inspection, a succinct equation, similar to the del Pezzo case, exists

$$f(t; X) = \frac{1 + (g-2)t + (g-2)t^2 + t^3}{(1-t)^4} = \sum_{n=0}^{\infty} \frac{t^n}{6}(2n+1)\Big((g-1)n^2 + (g-1)n + 6\Big), \tag{2.2}$$

where g is the genus of X.

In the special cases where the Fano threefold X is the product of dP_n with $\mathbb{P}^1$, the genus turns out to be $28 - 3n$. Whence, the number of generators of the moduli space is $30 - 3n = 3(10 - n)$; the 3 corresponds to the $\mathbb{P}^1$ factor, and the $10 - n$ refers to the dP_n factor.

3. Reconstructing the Vacuum Moduli Space

With a current want of an inverse algorithm, with or without the aid of dimer technology, it is difficult to systematically find the requisite quiver Chern-Simons theories whose moduli spaces are Calabi-Yau cones over the Fano threefolds listed above, a question certainly of considerable interest. Nevertheless, because the forward algorithm is now well established [20], one could explicitly check whether a certain ansatz theory indeed gives the correct moduli space. Therefore, with a combination of inspired guesses and systematic computer scans, one could hope to find some theories.

Nomenclature

In accordance with the notation of [14, 19], and emphasising the intimate relation between the $(3+1)$-dimensional gauge theory and the $(2+1)$-dimensional QCS, we denote the latter as follows: let the superpotential and matter content be that of the D3-brane world-volume theory for the Calabi-Yau threefold X, then we keep the same superpotential and quiver, but impose Chern-Simons levels $\vec{k}$, ordered according to a fixed choice for the nodes, while obeying the constraint [19, 20]

$$\sum_i k_i = 0, \qquad \mathrm{GCD}(k_i) = 1. \tag{3.1}$$

We subsequently run the forward algorithm, the resulting vacuum moduli space is now a Calabi-Yau fourfold and the QCS theory we will denote as $\widetilde{X}_{\vec{k}}$. Note, of course, that the actual 4-fold may be seemingly quite unrelated to X.

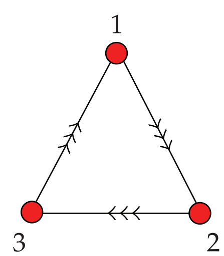
Furthermore, as always, we let X_{ij}^{a} denote the ath bifundamental field between nodes i and j, and let ϕ_i^a signify the ath adjoint field for the ith node.

3.1. Various Candidates

$\widetilde{dP_0}_{(1,-2,1)}$ and $\mathcal{B}_4$

The quiver and superpotential can be readily recalled from, for example, [27, 28] (cf. also this theory as a QCS from [13]); next, we can assign the Chern-Simons levels as $(1,-2,1)$, which indeed satisfies the constraint (3.1)

$$W = \epsilon_{\alpha\beta\gamma} X_{12}^{(\alpha)} X_{23}^{(\beta)} X_{31}^{(\gamma)}, \qquad \text{CS-levels} = (1,-2,1). \tag{3.2}$$



Running through the forward algorithm gives us the following charge matrix Q_t and toric diagram G_t:

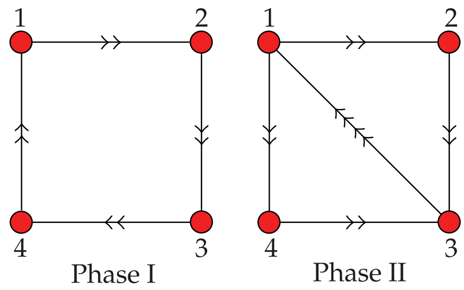
$$Q_t = \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}, \qquad G_t = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{3.3}$$

Now, take $\mathcal{B}_4$, or number 24, of the Fano list from Table 2, and consider the affine CY4 cone thereupon, by adding a row of 1s. One can readily check that upto reordering the columns, the two G_t matrices are explicitly related by a $PSL(4;\mathbb{Z})$ transformation. This means that the moduli spaces, as affine toric varieties, are isomorphic.

Phases of F_0

Next, we recall the well-known two phases of the (3 + 1)-dimensional theories for the CY3 over the zeroth Hirzebruch surface

$$W_{(F_0)_I} = \epsilon_{ij}\epsilon_{pq}X_{12}{}^{i}X_{23}{}^{p}X_{34}{}^{j}X_{41}{}^{q},$$
$$W_{(F_0)_{II}} = \epsilon_{ij}\epsilon_{mn}X_{12}^{i}X_{23}^{m}X_{31}^{jn} - \epsilon_{ij}\epsilon_{mn}X_{14}^{i}X_{43}^{m}X_{31}^{jn}\ .$$



(3.4)

There are two toric phases, the first having 8 fields, and the second 12.

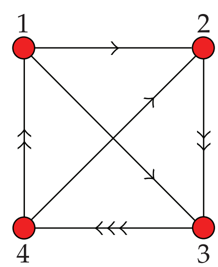
From these progenitors, we can obtain quite a few Calabi-Yau fourfold cones with judicious choices of CS levels. We list these in Table 3, running, in each case, the forward algorithm to the theory. The input is the superpotential and quiver of the indicated phase of F_0, together with the chosen Chern-Simons levels, and the output, the charge matrix Q_t and toric diagram G_t.

In this table, we have used the notation $\sim Cone(X)$ to mean that it is isomorphic, by an explicit $SL(4;\mathbb{Z})$ transformation of the toric diagrams (upto repetition and permutation of the vertices) G_t to the Calabi-Yau fourfold cone over the Fano threefold X. Note that the last row of G_t is always 1, this is a consequence of the Calabi-Yau condition. Furthermore, note that the second 2 rows for phase I, corresponding to the F-terms, decouple into diagonal form; this reflects the fact that the master space [52, 53] is the direct product of two conifolds. Moreover, the first row of the table, for the theory corresponding to $(\mathbb{P}^1)^{\times 3}$, has been obtained in [25].

$\widetilde{dP_1}$ *and* $\mathfrak{D}_1$

The theory for the cone over the dP_1 surface is again well known. We present it below (note that only two of the three bifundamental fields X_{34} group into an $SU(2)$ multiplet and the third is a singlet). Now, if we took the Chern-Simons levels as $(-1,-1,0,2)$, and combining with the standard theory

$$W = \epsilon_{ab}X_{13}X_{34}^{a}X_{41}^{b} + \epsilon_{ab}X_{42}X_{23}^{a}X_{34}^{b} + \epsilon_{ab}X_{34}^{3}X_{41}^{a}X_{12}X_{23}^{b},$$
$$\text{CS-levels} = (-1,-1,0,2),$$



(3.5)

Table 3: The two phases of the (3 + 1)-dimensional gauge theory for the cone over the zeroth Hirzebruch surface F_0 beget 4 new QCS theories in (2 + 1) dimensions, the moduli spaces for which are cones over 4 different Fano threefolds.

F_0	CS Levels $\vec{k}$	Q_t	G_t	$\sim Cone(X)$
I	$(1,1,-1,-1)$	$\begin{pmatrix} 1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\mathcal{C}_3$
I	$(-2,0,1,1)$	$\begin{pmatrix} 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\mathcal{C}_4$
I	$(-2,1,0,1)$	$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\mathcal{C}_5$
II	$(-2,0,1,1)$	$\begin{pmatrix} 0 & -2 & 0 & 0 & 1 & 1 & 2 & -2 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\mathcal{C}_4$
II	$(-2,1,0,1)$	$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\mathcal{C}_1$

then we find the charge and toric matrices to be

$$Q_t = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}, \qquad G_t = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{3.6}$$

and resulting moduli space to be $\mathfrak{D}_1$.

4. Outlook

In this short note, a prelude to [38], we have initiated the study of Fano threefolds in the context of M2-branes. In particular, we have presented the classification of all smooth toric Fano threefolds, the cones over which are Calabi-Yau fourfold singularities which the M2-branes could probe. We have computed some preliminary geometrical data, including such quantities as Hilbert series and global symmetries which have recently turned out to be important for the physics of these models.

These 18 spaces are direct analogues of the toric del Pezzo surfaces, which have been the subject of much investigation in the past decade in association with the construction of (3 + 1)-dimensional world-volume quiver gauge theories for D3-branes. It is self-evident that these spaces should be central to the study of (2 + 1)-dimensional quiver Chern-Simons theories.

For some of these we have identified, using the forward algorithm, the quiver theories whose mesonic moduli spaces are precisely as desired. Such a *prima facie* scan has produced 6 as moduli spaces of vacua, and they, as with all theories so far produced in the toric M2-brane scenario, obey the planar brane tiling/dimer model condition. It is our hope that systematically all gauge theories for the 18 spaces can be soon geometrically engineered and the corresponding tiling descriptions prescribed. These and many further details will appear in the companion work of [38].

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בהקדשה לואלרי ואֱלָה שמאירות אתחיי ,תמיד.

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