On the Fermionic Quasi-particle Interpretation in Minimal Models of Conformal Field Theory

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The conjecture that the states of the ferminonic quasi-particles in minimal conformal field theories are eigenstates of the integrals of motion to certain eigenvalues is checked and shown to be correct only for the Ising model.

I. INTRODUCTION

The Hilbert spaces of two dimensional conformal field theories [1] are described in terms of chiral algebras which act on a finite set of fields. The basis of these spaces are however not unique, which in turn, on the basis of the characters, leads to very interesting generalizations of the famous identities of Rogers-Schur and Ramanujan [2,3] appearing in number theory.

Character formulae for irreducible highest weight representations of the Virasoro algebra (*Vir*) exist in several alternative forms. The oldest, very often refered to as bosonic representation, are the formulae of Feigin and Fuchs and Rocha-Caridi [4,5], which directly incorporate the structure of the Null-vectors, i.e. divides out the invariant ideal. When interpreted as partition function one demands modular invariance of these expressions. In order to investigate this property it is most natural to re-express the characters in terms of Theta functions for which these transformations are well-known. Furthermore via the Jacobi triple identity it is possible to establish an easy relation to the currently ubiquitous quantum dilogarithm [6], which is very useful to carry out semi-classical limits. In order to understand the relation to massive integrable theories, or more precisely perturbed conformal field theories [7] more recent formulae, also known as fermionic representations are most suitable. These expressions posses a remarkable direct fermionic interpretation in terms of quasi-particles for the states, obeying Pauli's exclusion principle. The connection between these states and the spectrum of quasi-particle excitations, which arise from the Bethe Ansatz equations for the eigenvalues of the Hamiltonians, have recently been elaborated by the Stony Brook group [8–11] .

Whereas the former bosonic representations are in general unique, the latter are not, possibly indicating different relevant perturbations already at the conformal point of the theory. Ultimately one would like to construct these states explicitly in terms of cosets [12], in a sense we shall specify below.

In the present letter we will check the simple conjecture that the fermionic states are eigenvalues of the integrals of motion of the same form as in the continuum theory.

II. THE QUASI-PARTICLE INTERPRETATION

For the convenience of the reader and in order to establish our notation we shall briefly recall various equivalent expressions for the characters and then explain how they lead to an interpretation in terms of quasi-particles. First of all the bosonic representation [4,5] for the minimal models $\mathcal{M}(s,t)$ [1] with central charge $c = 1 - \frac{6(s-t)^2}{s\ t}$ and highest weight $h_{n,m} = \frac{(ns-mt)^2-(s-t)^2}{4\ s\ t}$ reads

$$\chi^{s,t}_{n,m}(q) = \frac{q^{-\frac{c}{24}}}{(q)_\infty} \sum_{k=-\infty}^{\infty} \left(q^{h_{n+2kt,m}} - q^{h_{n+2kt,-m}} \right) \quad . \quad (1)$$

Here we have used the standard abbreviation $(q)_m = \prod_{k=1}^{m} (1-q^k)$. Introducing the parameterization $q = e^{i\pi\tau}$, $k_\pm = \tau\pi(s\ n \pm m\ t)$, $k_0 = \tau s\ t$ and using the well known formula for one of the Theta functions $\Theta_3(p,\tau) = \sum_{k=-\infty}^{+\infty} q^{k^2} e^{ipk}$ one easily derives

$$\chi^{s,t}_{n,m}(q) = \frac{1}{\eta(q)} \left(q^{\frac{k_-^2}{4\ k_0 \tau \pi^2}} \Theta_3(k_-,k_0) - q^{\frac{k_+^2}{4\ k_0 \tau \pi^2}} \Theta_3(k_+,k_0) \right)$$

with $\eta(q)$ denoting Dedekind's Eta function. The relation to quantum dilogarithms [6] $S_q(p) = \prod_{l=1}^{\infty} (1 + e^{ip} q^{2l-1})$ is easily established by employing Jacobi's triple identity

$$\Theta_3(q,\tau) \ = \ S_q(p)\ S_q(-p)\ S_q(\pi\tau) \quad . \quad (2)$$

Now one is in a nice position to carry out semi-classical limits, by knowing that the quantum dilogarithm acquires classically the form of Roger's dilogarithm [16]

$$\lim_{\tau\to 0} S_q(\pi + p) \ = \ \exp\left(\frac{1}{2\pi i \tau}\ L_2(e^{ip}) \ + \ \mathcal{O}(\tau) \right) \quad . \quad (3)$$

For the fermionic representation there exist two versions, which are of slightly different nature when interpreted as partition functions. First

$$\chi^{\vec{m}}_{A,\vec{B}}(q) = \sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m} A \vec{m}^t + \vec{m}\cdot\vec{B}}}{(q)_{m_1} \dots (q)_{m_n}} \quad (4)$$

where A is a $N \times N$-matrix, with being the number of species, and $\vec{B}$ denotes a vector which needs to be specified for a particular theory and super-selection sector. The summation over $m_1, m_2, \ldots$ may be restricted in some way indicating that certain particles may only appear in conjunction with others. Choosing A to be the inverse of the Cartan matrix C, related to a particular simply laced Lie algebra, it was found in [9] that for A_n one obtains the Z_{n+1} invariant parafermionic theories, for the D_n case the $r = \sqrt{n/2}$ orbifold theories and for E_6, E_7, E_8 the tricritical three state Potts -, the tricritical Ising - and the Ising model, respectively. Different restrictions on the summation correspond in this case in general to different symmetries of the Dynkin diagram and different vectors $\vec{B}$ are related to different super-selection sectors. In [9] these expressions where found by means of Mathematica and to the knowledge of the authors explicit analytic proofs are still lacking for most of them. An exception is the character formula for the $h_{11} = 0$ sector of the E_8-theory, for which Warnaar and Pearce [13] utilized the connection between the Ising model and the dilute A_3-model in order to proof it.

Of different nature are the expressions for the minimal models $\mathcal{M}(s = l+2, t = l+3)$ which are entirely based on A_l

$$\chi^l_{n,m}(q) = \sum_{\vec{k}} \frac{q^{\frac{1}{4}\vec{k}C\vec{k}^t - \frac{1}{2}\vec{e}_{l+2-m}\cdot\vec{k} - \frac{1}{4}(m-n)(m-n-1)}}{(q)_{k_1}} \prod_{a=2}^{l} \begin{bmatrix} \frac{1}{2}\left(\vec{k}I_l - \vec{e}_n + \vec{e}_{l+2-m}\right)_a \\ k_a \end{bmatrix}_q \tag{5}$$

where the q deformed binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m (q)_{n-m}} \tag{6}$$

has been introduced and the sum is restricted to $\vec{k} \in (2\mathbb{Z})^n + (m-1)(\vec{e}_1 + \ldots \vec{e}_l) + (\vec{e}_{n-1} + \vec{e}_{n-3} \ldots) + (\vec{e}_{l+3-m} + \vec{e}_{l+5-m} \ldots)$. Furthermore $\vec{e}_j$ denotes the l-dimensional unit vector in the direction j and I_l is the incidence matrix $I_l = 2 - C_l$, of A_l. This formula was proven by Berkovich et. al. [14]. (4) and (5) only coincide for $c = \frac{1}{2}, \frac{4}{5}$. As shown by Kedem, Klassen, McCoy and Melzer both possess an interpretation in terms of fermionic quasi-particles. For this purpose one considers the characters as partition function Z

$$\chi \sim Z = \sum_{\text{states}} e^{-\frac{\text{E(states)}}{kT}} = \sum_{l=0}^{\infty} P(E_l) e^{-\frac{E_l}{kT}} \tag{7}$$

with T being as usual the Temperature, k Boltzmann's constant, E_l and $P(E_l)$ the energy and the degeneracy of the particular level l, respectively. Regarding the system as a gas of particles in a box of size L, L is thought to be large, one may quantize the possible momenta in units of $2\pi/L$. Assuming that there are n different species of particles one obtains this way a set of single particle momenta $p^a_{i_a}$ for the quasi-particles, characterized by a, the particle type and a non-negative integer i_a, which is subject to some restrictions depending on the particular model. The energy spectrum minus the ground state energy and the one for the related momenta may then be expressed as

$$E_l = \sum_{a=1}^{n} \sum_{i_a=1}^{m_a} e_a(p^a_{i_a}) \qquad p_l = \sum_{a=1}^{n} \sum_{i_a=1}^{m_a} p^a_{i_a}. \tag{8}$$

By definition if a many-body system obeys (8) in the infinite size limit, its spectrum is said to be of quasi-particle type. For the quasi-particles to be of fermionic nature one requires that one of the restrictions acquires the form of Pauli's exclusion principle

$$p^a_{i_a} \neq p^a_{j_a} \qquad \text{for } i_a \neq j_a \ . \tag{9}$$

Formally this is achieved by employing the well known formula from number theory, (refer for instance [3]) for the number of partitions $P_M(n,m)$ of a non-negative integer n into M distinct non-negative integers which are smaller than m

$$\sum_{n=0}^{\infty} P_M(n,m) q^n = q^{\frac{1}{2}M(M-1)} \begin{bmatrix} m+1 \\ M \end{bmatrix}_q \quad . \tag{10}$$

In case there is no upper limit, i.e. when m tends to infinity, we simply employ on the right hand side

$$\lim_{m\to\infty} \begin{bmatrix} m+1 \\ M \end{bmatrix}_q = \frac{1}{(q)_M} \quad . \tag{11}$$

The requirement of distinctiveness expresses here the fermionic nature of the quasi-particles in this Ansatz. Employing (10) in (4) and choosing $q = e^{-\frac{2\pi v}{kTL}}$, v being the velocity, it is straightforward to derive that the possible set of momenta is

$$p^a_{i_a} = \frac{2\pi}{L}\left(\left(A_{ab} - \frac{1}{2}\right) m_b + B_a + \frac{1}{2} + N_{i_a}\right) \tag{12}$$

where N_{i_a} is a set of non-negative distinct integers. Proceeding in the same way for (5) one ends up with some restrictions from above for the possible momenta for all particles except the first one. This feature distinguishes (5) from (4), since in the latter case the particles are more on the same footing, whereas in the former case particle one plays the dominant role. One obtains for the possible momenta, in units of $\frac{2\pi}{L}$, of the minimal models

$$p^1_{i_1} \in \left\{p_1^{min}(\vec{k}), p_1^{min}(\vec{k}) + 1, p_1^{min}(\vec{k}) + 2, \ldots\right\}$$
$$p^a_{i_a} \in \left\{p_a^{min}(\vec{k}), p_a^{min}(\vec{k}) + 1, \ldots, p_a^{max}(\vec{k})\right\}$$

with

$$\vec{p}^{min}(\vec{k}) = \frac{1}{2}\left(\vec{e}_1 + \vec{e}_2 + \ldots + \vec{e}_l - \vec{e}_{l+2-m} - \frac{\vec{k}\cdot I_l}{2}\right)$$
$$\vec{p}^{max}(\vec{k}) = \frac{1}{2}\left(\vec{e}_n - \vec{e}_1 - \vec{e}_2 - \ldots - \vec{e}_l + \frac{\vec{k}\cdot I_l}{2}\right)$$

In this way one may associate to each energy level some well defined set of fermionic quasi-particle momenta

$$|p_1^1, \ldots, p_1^{m_1}, \ldots, p_n^1, \ldots, p_n^{m_n}\rangle \ , \tag{13}$$

which are in one-to-one correspondence to the decomposition of the Hilbert space

$$\mathcal{H}_{h_{n,m}} = \bigoplus_{l=0}^{\infty} |h_{n,m}^l\rangle \tag{14}$$

in form of the irreducible representations of the Virasoro algebra

$$\begin{aligned} L_0|h_{n,m}\rangle &= h_{n,m}|h_{n,m}\rangle \\ L_k|h_{n,m}\rangle &= 0 \qquad \text{for} \quad k>0 \\ |h_{n,m}^l\rangle &= L_{-n_1}L_{-n_2}\ldots L_{-n_n}|h_{n,m}\rangle \qquad \text{for} \quad n_i > 0 \ . \end{aligned}$$

Here $l = n_1 + \ldots + n_n$ denotes the l^{th} level of the irreducible highest weight module with respect to the weight $h_{n,m}$. The question of how to construct these states explicitly in terms of cosets of the Kac-Moody algebra arises naturally, but is still an open problem [12]. Here we shall be less ambitious and only try to find some further properties of these quasi-particle states. An explicit construction will of course ultimately also allow to answer this question. For algebras based on $\hat{sl}(2)_k$ and $\hat{su}(k)_1$ such a constructions have been provided in terms of a spinon basis [15].

III. INTEGRALS OF MOTION

In the case of massive integrable models conserved charges serve as a very powerful tool. By assuming locality in the momentum space and the possibility of diagonalising them as one-particle asymptotic states, they may be employed, even without knowing their explicit form, to construct the scattering matrix of a purely elastic scattering theory [7]. For the massless models such infinite Abelian subalgebra of integrals of motions are known to exist in the enveloping algebra of the Virasoro algebra ($UVir$) [17]. The charge densities $T_{2k}(z)$ acquire unique expressions from the requirement of mutual commutativity and the assignment of a definite spin. For a chiral field in the plane this may be achieved algebraically by demanding the condition for a primary field $T_{2k}(z)$ with conformal dimension $2k$, ($k \in \mathbb{N}$)

$$[L_n, T_{2k}(z)] \ = \ z^{n+1}\frac{dT_{2k}(z)}{dz} \ + \ 2k\,(n+1)\,z^n T_{2k}(z) \tag{15}$$

only to be satisfied for the Möbius subalgebra, i.e. for $n = \pm 1, 0$. This means $T_{2k}(z)$ is asked to be a quasi-primary field. Regarding the map from the plane to the cylinder as a particular conformal transformation, i.e. $z = e^{\omega}$, one has a well defined procedure to obtain the spin densities defined on the cylinder, for instance $T_2^{cyl.}(\omega) = z^2 T(z) - \frac{c}{24}$. Integration of the charge densities will give the integrals of motion of spin value $2k-1$

$$I_{2k-1} \ = \ \int_0^{2\pi} \frac{d\omega}{2\pi} T_{2k}(\omega) \ . \tag{16}$$

The first of them read [17,18]

$$I_3 = 2\sum_{n=1}^{\infty} L_{-n}L_n \ + \ L_0^2 \ - \ \frac{2+c}{12}L_0 + k_3 \tag{17}$$

$$\begin{aligned} I_5 = &\sum_{n,m,l} :L_nL_mL_l:\delta_{n+m+l,0} + \frac{3}{2}\sum_{n=1}^{\infty} L_{1-2n}L_{2n-1} \\ &+\sum_{n=1}^{\infty}\left(\frac{11+c}{6}n^2 - \frac{c}{4} - 1\right)L_{-n}L_n \\ &-\frac{4+c}{8}L_0^2 + \frac{(2+c)(20+3c)}{24^2}L_0 + k_5 \ . \end{aligned} \tag{18}$$

Here k_3, k_5 are constants which depend on the superselection sector. It will turn out that for our purposes these constants have to be chosen differently than in [18]. $: \ :$ is the usual normal ordering prescription which arranges the operators L_n into an increasing sequence with respect to their mode index. We shall now employ these charges in order to give some characterization of the quasi-particle states. On the representation space of the Virasoro algebra they possess a well-defined action and one may compute explicitly their eigenvalues for each level

$$I_s|h_{n,m}^l\rangle \ = \ \lambda_{n,m}^{(s)}|h_{n,m}^l\rangle \ . \tag{19}$$

Denoting by $\mathcal{P}_{h_{n,m}^l}$ the particular set of momenta which correspond to the level l in the Verma module of $h_{n,m}^l$ the conjecture that the quasi-particle states are eigenstates of the integrals of motion arises naturally

$$I_s\,\mathcal{P} \ = \ \gamma^{(s)}_{\mathcal{P}_{h_{n,m}^l}}\ \mathcal{P} \ , \tag{20}$$

with $\mathcal{P} = |p_1^1, \ldots, p_1^{m_1}, \ldots, p_n^1, \ldots, p_n^{m_n}\rangle$. Drawing an analogy to the continuum theory, where the conserved charges act additively and diagonal on asymptotic one-particle states one may conjecture for the eigenvalues

$$\lambda^{(s)}_{h_{n,m}^l} \ = \ \gamma^{(s)}_{\mathcal{P}_{h_{n,m}^l}} = \sum_{a=1}^{N}\chi_a^{(s)}\sum_{j_a=1}^{m_a}\left(p_a^{j_a}\right)^s \ + \ \text{const} \ . \tag{21}$$

A. The Ising Model

We shall start by verifying this conjecture for the Ising Model, i.e. $c = \frac{1}{2}$, for which we expect it certainly to be true since it is known to be equivalent to a free fermion theory. In this case we have the following identities for the characters

$$q^{\frac{1}{48}} \chi^{3,4}_{1,1}(q) = \sum_{m=0}^{\infty} \frac{q^{2\,m^2}}{(q)_{2m}} \tag{22}$$

$$q^{\frac{1}{48}} \chi^{3,4}_{1,3}(q) = \sum_{m=0}^{\infty} \frac{q^{2\,m^2+2m+\frac{1}{2}}}{(q)_{2m+1}} \tag{23}$$

$$q^{\frac{1}{48}} \chi^{3,4}_{1,2}(q) = \sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m}{2}-\frac{1}{16}}}{(q)_m} \tag{24}$$

The results for the eigenvalue computation for the highest weight representation h_{11} are given in table I and II. The eigenvalues λ are computed explicitly by acting with the integrals of motion on the highest weight representation. We present the computation until level nine, since then the second Null-vector appears in the Verma modul. For the computation it is not necessary to divide out the complete invariant ideal, but only all descendants of L_{-1}, which is a Null-vector at level zero. This way one obtains of course more eigenvalues, but one may easily match them with the values of the appropriate γ. In principle the constants $\chi_1^{(s)}$ could have been fixed already after the first few levels and all higher levels serve as a consistency check. The constants k_3, k_5 were found to be different in the Ramond- and Neveu-Schwarz sector

$$k_3 = \frac{55}{3\cdot 16^2}\,\delta_{\frac{1}{16},h} \qquad k_5 = \frac{2161}{9\cdot 16^3}\,\delta_{\frac{1}{16},h} \quad . \tag{25}$$

For the constants χ we obtain

$$\chi_1^{(3)} \;=\; \frac{7}{6} \qquad \chi_1^{(5)} \;=\; \frac{143}{144} \quad . \tag{26}$$

Notice that the $\chi_1^{(s)}$ do not depend on h, i.e. they are universal constants independent of the super-selection sector.

B. The unitary minimal Models

We will now carry out a similar argumentation for the unitary minimal models. First we consider the 11-sector, for which (5) takes on a particular simple form

$$\chi^l_{1,1}(q) = \sum_{\vec{k}\in(2\mathbb{Z})^2} \frac{q^{\frac{1}{4}\vec{k}\cdot C\cdot\vec{k}^t}}{(q)_{l_1}} \prod_{a=2}^{l} \begin{bmatrix} \frac{1}{2}\left(\vec{k}I_l - \vec{e}_1\right)_a \\ k_a \end{bmatrix}_q \tag{27}$$

such that the minimal momentum becomes

l	d	$\lambda^{(3)}_{h^l_{1,1}}$	$\gamma^{(3)}_{\mathcal{P}_{h^l_{1,1}}}$
0	1	0	0
1	0	-	-
2	1	$\frac{49}{12}$	$\chi_1^{(3)}\frac{7}{2}$
3	1	$\frac{147}{8}$	$\chi_1^{(3)}\frac{63}{4}$
4	2	$(\frac{133}{6}, \frac{301}{6})$	$\chi_1^{(3)}(43, 19)$
5	2	$(\frac{1295}{24}, \frac{2555}{24})$	$\chi_1^{(3)}(\frac{365}{4}, \frac{185}{4})$
6	3	$(\frac{273}{4}, \frac{441}{4}, \frac{777}{4})$	$\chi_1^{(3)}(\frac{333}{2}, \frac{189}{2}, \frac{117}{2})$
7	3	$(\frac{2989}{24}, \frac{4753}{24}, \frac{7693}{24})$	$\chi_1^{(3)}(\frac{1099}{4}, \frac{679}{4}, \frac{427}{4})$
8	5	$(\frac{217}{3}, \frac{469}{3}, \frac{637}{3}, \frac{973}{3}, \frac{1477}{3})$	$\chi_1^{(3)}(422, 278, 182, 134, 62)$
9	5	$(\frac{1029}{8}, \frac{1953}{8}, \frac{2709}{8}, \frac{3969}{8}, \frac{5733}{8})$	$\chi_1^{(3)}(\frac{2457}{4}, \frac{1701}{4}, \frac{1161}{4}, \frac{837}{4}, \frac{441}{4})$

TABLE I. Eigenvalues of I_3 for the $h = 0$ sector

l	d	$\lambda^{(5)}_{h^l_{1,1}}$	$\gamma^{(5)}_{\mathcal{P}_{h^l_{1,1}}}$
0	1	0	0
1	0	-	-
2	1	$\frac{8723}{1152}$	$\chi_1^{(5)}\frac{61}{8}$
3	1	$\frac{74503}{768}$	$\chi_1^{(5)}\frac{1563}{16}$
4	2	$(\frac{60203}{24^2}, \frac{300443}{24^2})$	$\chi_1^{(5)}(\frac{421}{4}, \frac{2101}{4})$

TABLE II. Eigenvalues of I_5 for the $h = 0$ sector

Table III and IV show some values for the equivalent computation for the other super-selection sectors.

l	d	$\lambda^{(3)}_{h^l_{2,1}}$	$\gamma^{(3)}_{\mathcal{P}_{h^l_{2,1}}}$	$\lambda^{(5)}_{h^l_{2,1}}$	$\gamma^{(5)}_{\mathcal{P}_{h^l_{2,1}}}$
0	1	$\frac{7}{48}$	$\chi_1^{(3)}\frac{1}{8}$	$\frac{143}{4608}$	$\chi_1^{(5)}\frac{1}{32}$
1	1	$\frac{63}{16}$	$\chi_1^{(3)}\frac{27}{8}$	$\frac{3861}{512}$	$\chi_1^{(5)}\frac{243}{32}$
2	1	$\frac{875}{48}$	$\chi_1^{(3)}\frac{125}{8}$	$\frac{446875}{4608}$	$\chi_1^{(5)}\frac{3125}{32}$

TABLE III. Eigenvalues of I_3, I_5 for the $h = \frac{1}{2}$ sector

l	d	$\lambda^{(3)}_{h^l_{1,2}}$	$\gamma^{(3)}_{\mathcal{P}_{h^l_{1,2}}}$	$\lambda^{(5)}_{h^l_{1,2}}$	$\gamma^{(5)}_{\mathcal{P}_{h^l_{1,2}}}$
0	1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
1	1	$\frac{59}{48}$	$\chi_1^{(3)} + \frac{1}{16}$	$\frac{19}{18}$	$\chi_1^{(5)} + \frac{1}{16}$
2	1	$\frac{451}{48}$	$\chi_1^{(3)}\,8 + \frac{1}{16}$	$\frac{4585}{144}$	$\chi_1^{(5)}32 + \frac{1}{16}$

TABLE IV. Eigenvalues of I_3, I_5 for the $h = \frac{1}{16}$ sector

level	d	momentum 1	momentum 2
1	0	–	–
2	1	$p_1^1(2,0,0,\ldots) = \frac{1}{2}$	$p_2^1(2,0,0,\ldots) = \frac{3}{2}$
3	1	$p_1^1(2,0,0,\ldots) = \frac{1}{2}$	$p_2^1(2,0,0,\ldots) = \frac{5}{2}$
4	2	$p_1^1(2,0,0,\ldots) = \frac{1}{2}$ $p_1^1(2,0,0,\ldots) = \frac{3}{2}$	$p_2^1(2,0,0,\ldots) = \frac{7}{2}$ $p_2^1(2,0,0,\ldots) = \frac{5}{2}$

TABLE V. Fermionic states in the unitary minimal models

$$\vec{p}^{\ min}(k_1, k_2, \ldots, k_l) = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\\vdots\end{pmatrix} - \frac{1}{4}\begin{pmatrix}k_2\\k_1+k_3\\k_2+k_4\\k_3+k_5\\\vdots\end{pmatrix} \qquad (28)$$

This means in <u>all</u> unitary minimal models the structure of the first levels may be build up entirely from particle one alone. Compare table V for this.

To be more general we keep now in (17) also the variable in front of L_0 to be arbitrary, say k_3^0. By the same procedure as in the previous section the action of I_3 on these states leads to the following equations

$$\begin{aligned}4 + c + 2\,k_3^0 + k_3 &= \frac{7}{2}\chi_1^{(3)}\\ 17 + c + 3\,k_3^0 + k_3 &= \frac{63}{4}\chi_1^{(3)}\\ 34 + 6\,c + \sqrt{148 + 88c + 16c^2} + 4\,k_3^0 + k_3 &= 43\chi_1^{(3)}\\ 34 + 6\,c - \sqrt{148 + 88c + 16c^2} + 4\,k_3^0 + k_3 &= 19\chi_1^{(3)}\ .\end{aligned}$$

It turns out that this is in fact the only solution for these equations and hence (21) only holds for free fermions. In principle the assignment of the right hand side to the left hand side in the last two equations might have been reversed, but we expect to recover the Ising model as a particular case and therefore we have chosen the above relations. It is remarkable that when taking the values of (17) up to level 3 the above equation become true identities for all values of the central charge. Proceeding in the same way for (18), naming the constant in front of L_0 , k_5^0, leads to the following equations

$$\begin{aligned}\frac{91}{12} - \frac{19}{48}c + 2\,k_5^0 + k_5 &= \frac{61}{8}\chi_1^{(5)}\\ \frac{581}{6} - \frac{5}{24}c + 3\,k_5^0 + k_5 &= \frac{1563}{16}\chi_1^{(5)}\end{aligned}$$

Choosing k_5 to be zero in order to be able to recover the Ising model we obtain already from these two equations that in fact $c = \frac{1}{2}$ is the only solution. Hence the fact that for the I_3 integral of motion the third level allowed an arbitrary solution must be viewed as a coincidence.

IV. CONCLUSIONS

Similarly as in the last section we may proceed for the other minimal models. For instance we have carried out the equivalent computation for the $\mathcal{M}(2,5)$ (the Yang-Lee edge) model, also with the same negative answer. In conclusion we can say that, except for the Ising model, the integrals of motion do not act in the same way on the fermionic quasi-particle states as expected from the continuum theory. It would be very interesting to investigate whether these states are at all eigenstates of the integrals of motion.

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