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# The ultraviolet Behaviour of Integrable Quantum Field Theories, Affine Toda Field Theory

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**Abstract**

We investigate the thermodynamic Bethe ansatz (TBA) equations for a system of particles which dynamically interacts via the scattering matrix of affine Toda field theory and whose statistical interaction is of a general Haldane type. Up to the first leading order, we provide general approximated analytical expressions for the solutions of these equations from which we derive general formulae for the ultraviolet scaling functions for theories in which the underlying Lie algebra is simply laced. For several explicit models we compare the quality of the approximated analytical solutions against the numerical solutions. We address the question of existence and uniqueness of the solutions of the TBA-equations, derive precise error estimates and determine the rate of convergence for the applied numerical procedure. A general expression for the Fourier transformed kernels of the TBA-equations allows to derive the related Y-systems and a reformulation of the equations into a universal form.

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# 1 Introduction

More than twenty years ago Yang and Yang [1] introduced in a series of seminal papers a technique which allows to compute thermodynamic quantities for a system of bosons interacting dynamically via factorizable scattering. This method was generalized less than ten years ago by Al.B. Zamolodchikov [2] to a system of particles which interact dynamically in a relativistic manner through a scattering matrix which belongs to an integrable quantum field theory. This latter approach, usually referred to as thermodynamic Bethe ansatz (TBA), has triggered numerous further investigations [3-13]. The reason for this activity is twofold, on one hand the TBA serves as an interface between conformally invariant theories and massive integrable deformations of them and on the other hand it serves as a complementary approach to other methods.

The TBA-approach allows to extract various types of informations from a massive integrable quantum field theory once its scattering matrix is known. Most easily one obtains the central charge $c$ of the Virasoro algebra of the underlying ultraviolet conformal field theory, the conformal dimension $\Delta$ and the factor of proportionality of the perturbing operator, the vacuum expectation values of the energy-momentum tensor $\langle T^{\mu}_{\ \mu} \rangle$ and other interesting quantities. Thus, the TBA provides a test laboratory in which certain conjectured scattering matrices may be probed for consistency.

In addition, the TBA is useful since it provides quantities which may be employed in other contexts, like the computation of correlation functions. For instance the constant of proportionality, the dimension of the perturbing field and $\langle T^{\mu}_{\ \mu} \rangle$ may be used in a perturbative approach around the operator product expansion of a two point function within a conformal field theory [14]. $\langle T^{\mu}_{\ \mu} \rangle$ may also be used as an initial value for the recursive system between different n-particle form factors [14].

In order to obtain data beyond $c$ one has to investigate the behaviour of the scaling function $c(r)$ ($r$ is the inverse temperature times a mass scale), which may be viewed as the deformed central charge of the Virasoro algebra. It will be the central aim of this manuscript to determine this function. Certain statements in this context can be made in complete generality, but of course ultimately one has to specify some concrete models in order to be more explicit. An attractive choice for these models are affine Toda field theories [15], since they cover a huge class of theories due to their Lie algebraic formulation and permit therefore to extract many universal features. Ultimately they also allow a comparison against standard perturbation theory in the spirit of [16].

The first computation of such a scaling function was carried out in [3] for the scaling Lee-Yang model and scaling 3-state Potts model (minimal $A_2^{(1)}$-affine Toda field theory). Meanwhile there exist several computations of this kind [5, 6, 7, 11].

The general behaviour one finds is

$$c(r) = c_{\text{eff}} + f(r) + \sum_{n=1}^{\infty} c_n r^{2n(1-\Delta)} \tag{1}$$

where $c_{\text{eff}} = c - 24h'$ is the effective central charge, with $c$ being the usual conformal anomaly, i.e. the central charge of the Virasoro algebra, and $h'$ the lowest conformal dimension of the underlying conformal field theory. The $c_n$ are some constants and typically $f(r) \sim r^2$ or $f(r) \sim r^2 \ln(r)$. However, in [12, 13] a quite different behaviour was observed, namely that $f(r) \sim (const - \ln(r))^{-2}$, which was attributed therein to zero mode fluctuations. It is this kind of behaviour which we find for all affine Toda field theories (see equn. (83)). We like to stress that the constant *const* should be different from zero.

In our investigation we slightly modify and generalize an approach which was introduced originally in [12]. We also include in our analysis the possibility that the statistical interaction is of general Haldane type [17] and investigate the TBA-equations adapted to this situation [18]. We investigate in detail the expressions for the TBA-kernels and their Fourier transformed versions, from which we obtain a universal form for the relevant TBA-equations. We derive the related Y-systems. Furthermore, we address the question of the existence and uniqueness of the solutions of the TBA equations, derive error estimates and the rate of convergence for the numerical procedure applied. Our analytical considerations culminate with the derivation of a general expression for the scaling function valid for all affine Toda field theories in which the underlying Lie algebra is simply laced (83) and which depends in a universal manner only on the rank of the algebra, its Coxeter number and the effective coupling constant. We compare the general formulae against the numerical solutions for several explicit models, i.e. for the $A_1^{(1)}, A_2^{(1)}, A_2^{(2)}, A_3^{(1)}$ and $(G_2^{(1)}, D_4^{(3)})$-affine Toda field theories.

Our manuscript is organized as follows: In section 2 we recall the TBA-equation for general Haldane statistics and explain how it can be used to compute explicitly expressions for scaling functions. In section 3 we derive general expression for the approximated analytical expression for the solutions of the TBA-equations and for the scaling functions $c(r)$. In section 4 we concretely test our statements for a system involving the scattering matrix of affine Toda field theories. We compare the general approximated analytical expression with the numerical solution for some explicit models. In section 5 we address the question of the existence and uniqueness of the solutions of the TBA equations and derive error estimates and the rate of convergence. We state our conclusions in section 6.

## 2 The TBA-equations

We consider a multi-particle system involving $l$ different types of particles and assume that the two-particle scattering matrix $S_{ij}(\theta)$ (as a function of the rapidity $\theta$) together with the corresponding mass spectrum ($m_i$ is the mass of particle type $i$) have been determined. The $l$ coupled TBA-equations which describe such a system, based on the assumption that the dynamical interaction is characterized by the scattering matrices $S_{ij}(\theta)$ and whose statistical interaction is governed by general Haldane statistics [17] $g_{ij}$*, read [18]

$$r m_i \cosh\theta + \ln\left(1 - e^{-L_i(\theta,r)}\right) = \sum_{j=1}^{l} \left(\Phi_{ij} * L_j\right)(\theta, r) \quad . \tag{2}$$

The scaling parameter $r$ is the inverse temperature times a mass scale and we denote as usual the convolution of two functions $f$ and $g$ by $(f*g)(\theta) := 1/(2\pi)\int d\theta' f(\theta-\theta')g(\theta')$ and the kernel by

$$\Phi_{ij}(\theta) := -i\frac{d}{d\theta}\ln S_{ij}(\theta) - 2\pi g_{ij}\delta(\theta) = \varphi_{ij}(\theta) - 2\pi g_{ij}\delta(\theta) \quad . \tag{3}$$

We recall the important fact that the functions $L_i(\theta, r)$ are related to the ratio of densities $\rho_r^i$ of particles inside the system over the densities of $\rho_h^i$ of available states as $L_i(\theta, r) = \ln(1 + \rho_r^i/\rho_h^i)$. This implies that $L_i(\theta, r) \geq 0$, a property we will frequently appeal to. In addition we will make use of the fact that $L_i(\theta, r)$ is an even function in $\theta$. Once the system of coupled non-linear integral equations (2) is solved (usually this may only be achieved numerically) for the $l$ unknown functions $L_i(\theta, r)$, one is in a position to determine the scaling function

$$c(r) = \frac{6r}{\pi^2}\sum_{i=1}^{l} m_i \int_0^{\infty} d\theta L_i(\theta, r)\cosh\theta \quad , \tag{4}$$

which may be viewed as the off-critical central charge of the Virasoro algebra. We shall now adopt the method of [12] and instead of regarding the TBA-equation as an integral equation, we transform it into an infinite order differential equation. This is possible with the sole assumption that the Fourier transform of the function $\Phi_{ij}(\theta)$ can be expanded as a power series

$$\widetilde{\Phi}_{ij}(k) = \int_{-\infty}^{\infty} d\theta \Phi_{ij}(\theta) e^{ik\theta} = 2\pi\sum_{n=0}^{\infty}(-i)^n \eta_{ij}^{(n)} k^n \quad . \tag{5}$$

*The bosonic and fermionic statistics correspond to $g_{ij} = 0$ and $g_{ij} = \delta_{ij}$, respectively.

Then it is a simple consequence of the convolution theorem† that the integral equation (2) may also be written as an infinite order differential equation

$$rm_i \cosh\theta + \ln\left(1 - e^{-L_i(\theta,r)}\right) = \sum_{j=1}^{l}\sum_{n=0}^{\infty} \eta_{ij}^{(n)} L_j^{(n)}(\theta,r) \quad . \tag{6}$$

We introduced here the abbreviation $L_i^{(n)}(\theta,r) = (d/d\theta)^n L_i(\theta,r)$. The two alternative formulations of the TBA-equations serve different purposes. Its variant in form of an integral equation (2) is most convenient for numerical studies, whereas the differential equations (6) turn out to be useful for analytical considerations.

From a numerical point of view it is advantageous to make a change of variables and instead of the functions $L_i(\theta,r)$ use the so-called pseudo-energies $\epsilon_i(\theta,r)$ defined via the equations

$$\epsilon_i(\theta,r) := -\sum_j g_{ij} L_j(\theta,r) - \ln\left(1 - e^{-L_i(\theta,r)}\right) \ . \tag{7}$$

For fermionic and bosonic type of statistics the formulation of the TBA-equations in terms of pseudo-energies is most common. However, since (7) may obviously not be solved in general, it is more convenient to keep the TBA-equations in terms of the $L_i(\theta,r)$ for general Haldane type of statistics. Alternatively we may relate all types of statistics to the fermionic one in the following way. Considering a system which interacts statistically via $g_{ij}$, we can define the quantity $g'_{ij} = \delta_{ij} - g_{ij}$ and parameterize all $L-$functions by $L_i(\theta,r) =: \mathcal{L}_i(\theta,r) = \ln\left(1 + e^{-\varepsilon_i(\theta,r)}\right)$. Then the TBA-equations (2) may be rewritten as

$$\varepsilon_i(\theta,r) = rm_i \cosh\theta - \sum_{j=1}^{l}\left[\left((\varphi_{ij} + 2\pi g'_{ij}\delta) * \mathcal{L}_j\right)(\theta,r)\right] . \tag{8}$$

It should be kept in mind that $\varepsilon_i(\theta,r)$ ($\neq \epsilon_i(\theta,r)$!) is now a formal parameter and, except in the fermionic case, it is not related to the ratio of particle densities $\rho_r^i/\rho_h^i$ in the characteristic way as a distribution function associated to the relevant statistics, i.e. $\epsilon_i(\theta,r)$ obtained as a solution of the equations (7).

# 3 The ultraviolet Limit

We shall now analytically investigate the behaviour of the scaling function in the ultraviolet limit, i.e. $r$ is going to zero. For this purpose we generalize the procedure of Zamolodchikov [12], which leads to approximated analytical expressions. We attempt to keep the discussion model independent and free of a particular choice

† $(f * g)(\theta) = 1/(2\pi)^2 \int dk \widetilde{f}(k)\widetilde{g}(k) e^{-ik\theta}$

of the statistical interaction as long as possible. Introducing the quantity $\hat{L}_i(\theta) := L_i(\theta - \ln(r/2), r)$ and performing the shift $\theta \to \theta - \ln(r/2)$, the TBA-equations (6) acquire the form

$$m_i e^\theta + \ln\left(1 - e^{-\hat{L}_i(\theta)}\right) = \sum_{j=1}^{l} \sum_{n=0}^{\infty} \eta_{ij}^{(n)} \hat{L}_j^{(n)}(\theta) \ . \tag{9}$$

Here we have neglected the terms proportional to $e^{2\ln(r/2)-\theta}$, under the assumption that $2\ln(r/2) \ll \theta$. Obviously the $r$-dependence has vanished, such that the $\hat{L}_i(\theta)$ are $r$-independent. The equation for the scaling function (4) becomes, under similar manipulations and the neglect of similar terms as in the derivation of (9),

$$c(r) = \frac{6}{\pi^2} \sum_{i=1}^{l} m_i \int_{\ln(r/2)}^{\infty} d\theta \hat{L}_i(\theta) e^\theta \quad . \tag{10}$$

In analogy to the procedure of [12], we consider now the so-called "truncated scaling function"

$$\hat{c}(r, r') = \frac{6}{\pi^2} \sum_{i=1}^{l} m_i \int_{r'}^{\infty} d\theta \hat{L}_i(\theta) e^\theta \quad , \tag{11}$$

which obviously coincides with $c(r)$ for $r' = \ln(r/2)$. At this point we make several assumptions:

i) The functions $\hat{L}_i(\theta)$ obey the equations (9).

ii) In the power series expansion (5) the coefficients are symmetric in the particle type indices, i.e. $\eta_{ij}^{(n)} = \eta_{ji}^{(n)}$.

iii) All odd coefficients vanish in (5), i.e. $\eta_{ij}^{(2n+1)} = 0$.

iv) The asymptotic behaviour of the function $\hat{L}_i(\theta)$ and its derivatives read

$$\lim_{\theta\to\infty} \hat{L}_i^{(n)}(\theta) = 0 \qquad \text{for } n \geq 1, 1 \leq i \leq l \quad , \tag{12}$$

$$\lim_{\theta\to\infty} e^\theta \hat{L}_i(\theta) = 0 \qquad \text{for } 1 \leq i \leq l \quad . \tag{13}$$

Assumption ii) is guaranteed when the two-particle scattering matrix is parity invariant and the statistical interaction is symmetric in particle type, $g_{ij} = g_{ji}$. The requirement iii) puts of course constraints on the scattering matrices for given statistics, or vice versa. It will turn out in the next section, that for fermionic statistics it is satisfied by all scattering matrices of interest to us. The asymptotic behaviour iv) will be verified in retrospective, that is all known numerical solutions exhibit this kind of asymptotic behaviour.

Under these assumptions the truncated scaling function may also be written as

$$\hat{c}(r,r') = \frac{3}{\pi^2}\sum_{i,j=1}^{l}\left(\sum_{n=1}^{\infty}\eta_{ij}^{(2n)}\sum_{k=1}^{2n-1}(-1)^{k+1}\hat{L}_i^{(k)}(r')\hat{L}_j^{(2n-k)}(r') + \eta_{ij}^{(0)}\hat{L}_i(r')\hat{L}_j(r')\right)$$
$$-\frac{6}{\pi^2}\sum_{i=1}^{l}\left(L(1-e^{-\hat{L}_i(r')}) + \frac{\hat{L}_i(r')}{2}\ln(1-e^{-\hat{L}_i(r')}) + m_i e^{r'}\hat{L}_i(r')\right). \quad (14)$$

Here $L(x)=\sum_{n=1}^{\infty}x^n/n^2+1/2\ln(x)\ln(1-x)$ denotes Rogers dilogarithm (e.g. [19]). The equality (14) is most easily derived when considering the differential equation which results from (11)

$$\frac{\partial\hat{c}(r,r')}{\partial r'} = -\frac{6}{\pi^2}e^{r'}\sum_{i=1}^{l}m_i\hat{L}_i(r') \quad . \quad (15)$$

One may now verify by direct substitution‡ that (15) is solved by (14) when i)-iii) hold. The constant of integration is fixed by the property $\hat{c}(r,\infty)=0$, such that (14) is exact if iv) holds.

### 3.1 The extreme Limit

One of the best known outcomes of the TBA-analysis is the fact that in the extreme ultraviolet limit the scaling function becomes the effective central charge of the underlying ultraviolet conformal field theory, i.e. $\lim_{r\to 0}c(r)=c_{\text{eff}}$. In order to carry out this limit, we note that the assumptions (12) and (13) also imply

$$\lim_{r,\theta\to 0}L_i^{(n)}(\theta,r)=0 \qquad \text{for } n\geq 1\,,1\leq i\leq l \quad , \quad (16)$$

such that equations (6) become a set of coupled non-linear equations for the $l$ constants $L_i(0,0)$

$$\ln\left(1-e^{-L_i(0,0)}\right)=\sum_{j=1}^{l}\eta_{ij}^{(0)}L_j(0,0) \quad . \quad (17)$$

Due to the fact that $L_i(\theta,r)$ is positive, we deduce that these equations admit physical solutions when $\eta_{ij}^{(0)}\leq 0$. For a given S-matrix this condition will restrict possible statistical interactions. With the help of the solutions of (17) we recover from (14) the well known formula for the extreme ultraviolet limit

$$\lim_{r\to 0}c(r)=c_{\text{eff}}=\frac{6}{\pi^2}\sum_{i=1}^{l}L(1-e^{-L_i(0,0)}) \quad . \quad (18)$$

These equations have been analyzed extensively in the literature for various models [3, 5, 20, 10] and as we demonstrated, also hold for general (Haldane) type of statistics.

‡The identity $\int_0^x dt\ln(1-e^{-t})=L(1-e^{-x})+x/2\ln(1-e^{-x})$ is useful in this context.

## 3.2 Next leading Order

In order to keep the next leading order in the ultraviolet limit we proceed as follows: Under the assumption that the symmetry property of $\hat{L}_i(\theta)$ still holds at this point of the derivation, we may neglect in (9) also the term $e^\theta$. Assuming that $\hat{L}_i(\theta)$ is large and (12) holds, the TBA-equation in the ultraviolet limit may be approximated by

$$\sum_{j=1}^{l}(\eta_{ij}^{(2)}\hat{L}_j^{(2)}(\theta)+\eta_{ij}^{(0)}\hat{L}_j(\theta))+e^{-\hat{L}_i(\theta)}=0 \quad . \tag{19}$$

Unfortunately this equation may not be solved analytically in its full generality. However, choosing now the statistics in such a way that the $\eta_i^{(0)}=\sum_{j=1}^{l}\eta_{ij}^{(0)}=0$ we may solve (19). In most cases we shall be considering in the following, this condition implies that we have fermionic type of statistics. For this case the solution of (19) reads

$$\hat{L}_i(\theta)=\ln\left(\frac{\sin^2\left(\alpha_i\left(\theta-\beta_i\right)\right)}{2\alpha_i^2\eta_i^{(2)}}\right)+\ln\left(\frac{\cos^2\left(\tilde{\alpha}_i\left(\theta-\tilde{\beta}_i\right)\right)}{2\tilde{\alpha}_i^2\eta_i^{(2)}}\right) \quad , \tag{20}$$

with $\alpha_i,\beta_i,\tilde{\alpha}_i,\tilde{\beta}_i$ being the constants of integration and $\eta_i^{(2)}=\sum_{j=1}^{l}\eta_{ij}^{(2)}$. Note that the assumption $\eta_i^{(2)}\neq 0$ is not always guaranteed below. We will discard the second term in the following w.l.g. Under the same assumptions the expression for the truncated scaling function (14) becomes

$$\hat{c}(r,r')=l+\frac{3}{\pi^2}\sum_{i=1}^{l}\left(\eta_i^{(2)}\left(\hat{L}_i^{(1)}(r')\right)^2-2e^{-\hat{L}_i(r')}\right) \quad . \tag{21}$$

Substitution of the solution (20) into (21) yields

$$\hat{c}(r,r')=l-\frac{12}{\pi^2}\sum_{i=1}^{l}\eta_i^{(2)}\alpha_i^2 \quad . \tag{22}$$

Notice that this expression for the truncated effective central charge is independent of the constants $\beta_i$ and $r'$. We use the latter property to argue that in fact the r.h.s. of (22) corresponds to the scaling function $c(r)$. Invoking now also the property $\hat{L}_i(\theta)=\hat{L}_i(2\ln(r/2)-\theta)$ we obtain the additional relations

$$\alpha_i=\frac{n\pi}{2(\beta_i-\ln(r/2))} \quad , \tag{23}$$

with $n$ being an odd integer, which we choose to be one. At the moment we do not have any further argument at hand in order to fix the remaining constant. Therefore we have

$$L_i(\theta,r) \;=\; \ln\left(\frac{\cos^2\left(\alpha_i\theta\right)}{2\alpha_i^2\eta_i^{(2)}}\right) \tag{24}$$

$$
\begin{aligned}
L_i^{(1)}(\theta,r) &= -2\alpha_i \tan(\alpha_i\theta) & (25)\\
L_i^{(2)}(\theta,r) &= -2\alpha_i^2/\cos^2(\alpha_i\theta) & (26)\\
L_i^{(3)}(\theta,r) &= -4\alpha_i^3 \tan(\alpha_i\theta)/\cos^2(\alpha_i\theta) & (27)\\
L_i^{(n)}(\theta,r) &\sim \alpha_i^n \quad . & (28)
\end{aligned}
$$

Using the fact that $\alpha_i$ tends to zero for small $r$, the equations (25)-(28) demonstrate the consistency with the assumption that the derivatives of $L_i(\theta,r)$ with respect to $\theta$ are negligible, i.e. equation (16). Closer inspection shows that for given $r$ the series build from the $L_i^{(n)}(\theta,r)$ starts to diverge at a certain value of $n$. Since (24) is not exact this does not pose any problem, but one should be aware of it. The scaling function becomes in this approximation

$$
c(r) = l - 3 \sum_{i=1}^{l} \frac{\eta_i^{(2)}}{(\beta_i - \ln(r/2))^2} \quad . \tag{29}
$$

As already pointed out there does not seem to be any argument in this approach which allows to fix the constant $\beta_i$. However, we will present below a natural guess and also resort to numerical data to fix it.

In order to perform more concrete computations one has to specify a particular model at this point.

# 4 Affine Toda Field Theory

Affine Toda field theories [15] constitute a huge, important and well studied class of relativistically invariant integrable models in 1+1 dimensions. The exact two-particle scattering matrices of all affine Toda field theories with real coupling constant were constructed on the base of the bootstrap principle [21] and its generalized version [26]. Various cases have been checked perturbatively. The theories exhibit an entirely different behaviour depending on whether they are self-dual or not. Here duality has a double meaning, on one hand it refers to the invariance of the algebra under the interchange of roots and co-roots (i.e. $A_n^{(1)}, A_{2n}^{(2)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$) and on the other hand it refers to the strong-weak duality in the coupling constant.

## 4.1 The S-matrix

### 4.1.1 Simply laced Lie algebras

We recall now some well known facts about the two particle scattering matrix and present some new features which will be important for our analysis below. For theories related to simply laced Lie algebras the two particle S-matrices [21, 22, 23] involving the $r$ different types of particles may be furnished into the universal form

$$S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q - \frac{c(i)+c(j)}{2} \right\}_{\theta}^{-\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_j} , \qquad 1 \leq i,j \leq r = \text{rank}\mathbf{g} . \tag{30}$$

The building blocks $\{x\}_\theta$ may be expressed in terms of hyperbolic functions, albeit it will be most convenient for our purposes to use their integral representation

$$\{x\}_\theta = \exp \int_0^\infty \frac{dt}{t \sinh t} \, f_{x,B}(t) \sinh\left(\frac{\theta t}{i\pi}\right) \tag{31}$$

with

$$f_{x,B}(t) = 8 \sinh\left(\frac{tB}{2h}\right) \sinh\left(\frac{t}{2h}(2-B)\right) \sinh\left(\frac{t}{h}(h-x)\right) . \tag{32}$$

We adopt here the notations of [23] and denote the Coxeter number by $h$, fundamental weights by $\lambda_i$, the colour values related to the bicolouration of the Dynkin diagram by $c(i)$ and a simple root $\alpha_i$ times $c(i)$ by $\gamma_i$. The Coxeter element is chosen to be $\sigma = \sigma_- \sigma_+$, where the elements $\sigma_\pm = \prod_{i \in \Delta_\pm} \sigma_i$ are introduced, with $\sigma_i$ being a Weyl reflections and $\Delta_\pm$ the set of simple roots with colour values $c(i) = \pm 1$, respectively. The effective coupling constant $B(\beta) = (\beta^2/2\pi)/(1+\beta^2/4\pi)$ depends monotonically on the coupling constant $\beta$ and takes values between 0 and 2.

Many of the properties of (30) are very well documented in the literature [21, 22, 23] and we will therefore only concentrate on those which are relevant for our investigations. One of the remarkable features is the strong-weak duality in the coupling constant $\beta$, which is a particular example for one of the dualities which are currently ubiquitous in string theoretical investigations. Whilst in the latter context duality serves as a powerful principle, we will frequently employ it as a simple consistency check on various expressions which arise during our computations. The strong-weak duality manifests itself by the fact that each individual building block, and consequently the whole S-matrix, possesses the symmetry $B \to 2 - B$. A further check, which we wish to pursue from time to time for consistency reasons, is taking the coupling constant to zero, i.e. $B(0) = 0$ or equivalently $B(\infty) = 2$, such that we obtain a free theory with $S_{ij}(\theta) = 1$. Also we want to compare with some known results for the so-called minimal part[§] of the scattering matrix (30), which is formally obtained by taking the limit $B \to i\infty$[¶].

It will be important below to recall that the block $\{1\}_\theta$ only occurs (with power one) in the scattering matrix between two particles of the same type, i.e. in $S_{ii}(\theta)$,

[§] This part satisfies by itself the bootstrap equations and the additional factor is of a CDD-nature, meaning that is does not produce any relevant poles inside the physical sheet.

[¶] For the blocks $\{x\}_\theta$ in form of hyperbolic functions this limit is trivial. For the integral representation (31 ) it requires a bit more effort, but is easily verified with the help of the Riemann-Lebesgue theorem (If $g(x) \in L_1(-\infty, \infty)$ then $\lim_{t \to \pm\infty} \int_{-\infty}^{\infty} g(x) e^{-itx} dx = 0$).

which implies for the S-matrices (30) $S_{ij}(0) = (-1)^{\delta_{ij}}$. A further relation which exhibits the occurrence of particular blocks is

$$S_{ij}(\theta) = \{2\}_\theta^{I_{ij}} \prod_{q=2}^{h-2} \left\{2q - \frac{c(i)+c(j)}{2}\right\}_\theta^{-\frac{1}{2}\lambda_i\cdot\sigma^q\gamma_j}, \qquad c(i) \neq c(j)\,. \tag{33}$$

Here $I_{ij}$ is the incidence matrix of the Lie algebraic Dynkin diagram, i.e. twice the unit matrix minus the Cartan matrix $K$. Since relation (33) will be important for our analysis and does not seem to appear in the literature, we will briefly prove it here. We make use of the interrelation between the incidence matrix and particular elements of the Weyl group and the action of these elements on simple roots [24, 23]

$$\sum_{j=1}^{r} I_{ij}\lambda_j = (\sigma_- + \sigma_+)\lambda_i \quad \text{and} \quad \sigma_{c(i)}\alpha_j = -\sigma^{(c(j)-c(i))/2}\alpha_j\ . \tag{34}$$

Taking the inner product of the first equation in (34) with $\alpha_j$ and the subsequent application of the second relation on the r.h.s. yields

$$I_{ij} = \lambda_i\cdot\sigma_{-c(j)}\alpha_j + \lambda_i\cdot\sigma_{c(j)}\alpha_j = -\lambda_i\cdot\alpha_j - \lambda_i\cdot\sigma^{c(j)}\alpha_j\ . \tag{35}$$

Therefore, by the orthogonality of fundamental weights and simple roots and the symmetry in $i$ and $j$, it follows directly that

$$\lambda_i\cdot\sigma^{\pm1}\gamma_j = \mp I_{ij}\,, \qquad c(i) \neq c(j)\quad . \tag{36}$$

Noting further that $\{x\}_\theta = \{2h-x\}_\theta^{-1}$, we may extract the block $\{2\}_\theta$ with power $I_{ij}$ from the product in (30) and obtain (33).

Remarkably one may also carry out the product in (30) and obtain a closed expression for $S$ in the form

$$S_{ij}(\theta) = \exp\left(i\int_0^\infty \frac{dt}{t}\left(f_{h+h\pi/(2t)),B}(t)\left(2\cosh\frac{\pi t}{h} - I\right)_{ij}^{-1} - 2\delta_{ij}\right)\sin\left(\frac{\theta t}{\pi}\right)\right)\ . \tag{37}$$

The Lie algebraic quantities involved in this identity, i.e. $I$ and $h$, are more easily accessible than the orbits of the Coxeter elements, however, this is at the cost that the singularity structure is less transparent. Formula (37) coincides with equation (5.2) in [25] (up to a factor $(-1)^{\delta_{ij}}$), once the general expression in there is reduced to the simply laced case.

In the course of our argumentation we will also employ the identity

$$S_{ij}\left(\theta - \frac{i\pi}{h}\right)S_{ij}\left(\theta + \frac{i\pi}{h}\right) = \prod_{l=1}^{r} S_{il}(\theta)^{I_{lj}} \qquad \theta \neq 0\ , \tag{38}$$

which was derived first in [9].

### 4.1.2 Non-simply laced Lie algebras

Once one turns to the consideration of affine Toda field theories related to non-simply laced Lie algebras many of the universal features which one observes for simply laced Lie algebras cease to be valid. For instance for the simply laced Lie algebras it is known that the singularities of (30) inside the physical sheet do not depend on the coupling constant and remarkably all masses renormalise with an overall factor. These latter behaviours dramatically change once the related Lie algebras are taken to be non-simply laced (except $A_{2n}^{(2)}$ which is also self-dual). In fact in these cases one can not associate anymore a unique Lie algebra to the quantum field theory, but a dual pair related to each other by the interchange of roots and co-roots. General formulae similar to (30), which are valid for all non-simply Lie laced algebras have not been constructed yet. However, one can still construct S-matrices [26, 27]

$$S_{ij}(\theta) = \prod_{x,y} [x, y]_\theta \tag{39}$$

out of some universal building blocks which generalize (31)

$$[x, y]_\theta = \exp \int_0^\infty \frac{dt}{t \sinh t} \; g_{x,y,B}(t) \sinh\left(\frac{\theta t}{i\pi}\right) \; . \tag{40}$$

In distinction to the simply laced case, the variable $x$ may now become non-integer, $y$ is $0, 1/2, 1$ or $-1/4$ and

$$g_{x,y,B}(t) = 8 \sinh\left(\frac{tB}{2H}(1-2y)\right) \sinh\left(\frac{t}{2H}(2-B)\right) \sinh\left(\frac{t}{H}(H-x)\right) \quad . \tag{41}$$

The Coxeter number $h$ has been replaced by a "floating Coxeter number " $H$, in the sense that it equals the Coxeter number of one of the two dual algebras in the weak limit and the other in the strong limit. The singularities of (39) now depend on the coupling constant which leads to a modification of the bootstrap as explained in [26]. The shift of the masses resulting from the renormalization procedure can not be compensated anymore by an overall factor.

In contrast to the simply laced case, the control over the set in which $x$ and $y$ take their values is slightly different than in (30). Whereas in (30) the powers of the blocks may be computed directly from the Lie algebraic quantities, in the non-simply laced case they are obtained solving at first some recursive equations and thereafter extracting the powers from a generating function [25]. There exists however a generalization of (37) which also includes the non-simply laced case. In this paper we will not treat the completely generic case and shall be content with treatment of one particular case in order to exhibit the difference towards the simply laced case, such that (39) and (40) are sufficient for our purposes.

## 4.2 The TBA-kernels

Given the S-matrices we are now in a position to compute the relevant quantities for our analysis, i.e. the kernels appearing in the TBA-equations (2) and the Fourier coefficients in equation (5).

### 4.2.1 Simply laced Lie algebras

Taking the logarithmic derivative of the phase of the scattering matrix (30) the TBA-kernels (without the statistics factor) are easily computed to

$$\varphi_{ij}(\theta) = -\frac{1}{2}\sum_{q=1}^{h}(\lambda_i \cdot \sigma^q \gamma_j)\omega_{2q-(c(i)+c(j))/2}(\theta) \quad , \tag{42}$$

where

$$\begin{aligned} \omega_x(\theta) &= -i\frac{d}{d\theta}\ln\{x\}_\theta = \frac{1}{\pi}\int_0^\infty \frac{dt}{\sinh t}\; f_{x,B}(t)\cos\left(\frac{\theta t}{\pi}\right), & (43)\\ &= \gamma_{x-1}(\theta) + \gamma_{x+1}(\theta) - \gamma_{x+B-1}(\theta) - \gamma_{x-B+1}(\theta)\,. & (44) \end{aligned}$$

We introduced the function $\gamma_x(\theta) = \sin(\pi x/h)/(\cos(\pi x/h) - \cosh\theta)$ and set $\gamma_0(\theta) = 0$. Depending on the context either the variant (43) or (44) turn out to be more convenient. The Fourier transform is most easily evaluated when we use the integral representation (43) for each block

$$\begin{aligned} \widetilde{\omega}_x(k) &= -\frac{\pi\, f_{x,B}(\pi k)}{\sinh \pi k} & (45)\\ &= \widetilde{\gamma}_{x-1}(k) + \widetilde{\gamma}_{x+1}(k) - \widetilde{\gamma}_{x+B-1}(k) - \widetilde{\gamma}_{x-B+1}(k), & (46) \end{aligned}$$

where $\widetilde{\gamma}_x(k) = 2\pi\sinh[(x/h-1)\pi k]/\sinh(\pi k)$. Therefore we have

$$\widetilde{\varphi}_{ij}(k) = -\frac{1}{2}\sum_{q=1}^{h}(\lambda_i \cdot \sigma^{q+(c(j)-1)/2}\gamma_j)\widetilde{\omega}_{2q+(c(j)-c(i))/2-1}(k)\ . \tag{47}$$

This means, that just like the S-matrix itself, we may express the TBA-kernels and their Fourier transformed versions in terms of general blocks. The difference is that now instead of the powers of the blocks their pre-factors are determined by orbits of the Coxeter element. Notice also that the property $\{x\}_\theta = \{2h+x\}_\theta$ guaranteed that in (30) the products $\prod_{q=1}^{h}$ and $\prod_{q=0}^{h-1}$ are equivalent. Since now $\widetilde{\gamma}_x(k) \neq \widetilde{\gamma}_{2h+x}(k)$, we have to take the sum over the appropriate range. This is the reason why $q$ is shifted in (47). Remarkably for the Fourier transformed kernels it is possible to evaluate the whole sum over $q$ in (47) in an indirect way. We obtain the universal form

$$\widetilde{\varphi}_{ij}(k) = 2\pi \sum_{l=1}^{r} \left( I - 2\cosh\frac{\pi}{h}k \right)^{-1}_{il} \left( I - 2\cosh\frac{\pi}{h}k(1-B) \right)_{lj} \tag{48}$$

$$= 8\pi \sinh\left(\frac{(B-2)\pi k}{2h}\right) \sinh\left(\frac{B\pi k}{2h}\right) \left(2\cosh\frac{\pi k}{h} - I\right)^{-1}_{ij} - 2\pi\delta_{ij} \; . \tag{49}$$

A similar matrix identity has turned out to be extremely useful in the investigation [4] of the TBA-kernel for the minimal part of the scattering matrix (30). Equation (48) coincides with an identity quoted in [13] once the roaming parameter therein is chosen in such the way that the resonance scattering matrices take on the form of (30). Taking the inverse Fourier transform of (49) we obtain after an intgeration with respect to $\theta$ the integral representation (37) for the scattering matrix.

Since the identity (48) is by no means obvious, in particular with regard to (47), we will now provide a rigourous proof of it. Essentially we have to use the Fourier transformed version of the relation (38) for this purpose. Shifting the Fourier integral into the complex plane yields

$$\lim_{\varrho\to\infty} \oint_{\mathcal{C}_\varrho^\pm} d\theta \varphi_{ij}(\theta) e^{ik\theta} = \widetilde{\varphi}_{ij}(k) - \mathcal{P}\int d\theta \varphi_{ij}\left(\theta \pm i\frac{\pi}{h}\right) e^{ik(\theta \pm i\pi/h)} \tag{50}$$

$$= 2\pi\delta_{ij} e^{\mp\pi B k/h} - \pi I_{ij} e^{\mp\pi k/h} \, . \tag{51}$$

Here the contours $\mathcal{C}_\varrho^\pm$ are depicted in figure 1. $\mathcal{P}$ denotes the Cauchy principal value. The poles inside the contours are collected by considering (42) and (44). Obviously coupling constant dependent poles may only result from $\omega_1(\theta)$ and $\omega_{2h-1}(\theta)$, whilst poles directly on the contours at $\theta = \pm i\pi/h$ may only originate from $\omega_2(\theta)$ and $\omega_{2h-2}(\theta)$. From the statements made in subsection 4.1.1., it follows that the former blocks may only occur when $i = j$ and from equation (33) we infer that the pre-factor of $\omega_2(\theta)$ is $I_{ij}$. The relevant residues are computed easily and by noting further that the singularities directly on the contour count half, we have established (51).

Acting now with $-i$ times the logarithmic derivative on the identity (38), multiplying with $\exp(ik\theta)$ and integrating thereafter with respect to $\theta$ we obtain

$$\mathcal{P}\int d\theta \left( \varphi_{ij}(\theta + i\pi/h) + \varphi_{ij}(\theta - i\pi/h) \right) e^{ik\theta} = \sum_{l=1}^{r} I_{il} \widetilde{\varphi}_{lj}(k) \; . \tag{52}$$

On the other hand the l.h.s. of (52) may be computed alternatively from the right hand sides of (50) and (51), such that we obtain

$$\sum_{l=1}^{r} I_{il} \widetilde{\varphi}_{lj}(k) = 2\widetilde{\varphi}_{ij}(k) \cosh\frac{\pi}{h}k + 2\pi I_{ij} - 4\pi\delta_{ij} \cosh\frac{\pi}{h}k(1-B) \; , \tag{53}$$

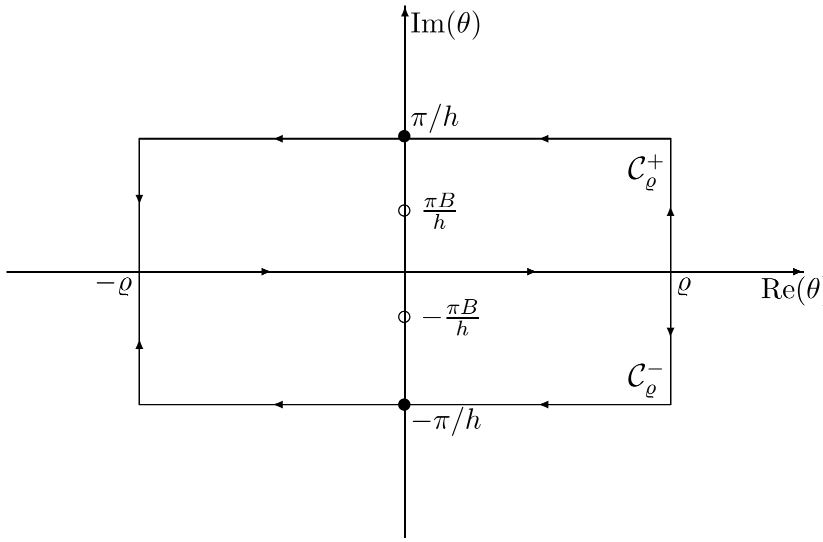


Figure 1: The contours $\mathcal{C}_\varrho^\pm$ in the complex $\theta$-plane. The bullets $\bullet$ belong to poles resulting from $\omega_2(\theta)$ and the open circles $\circ$ to poles of $\omega_1(\theta)$.

and therefore (48).

When we take the limit $B \to i\infty$, the coupling constant dependent poles move outside the contours $\mathcal{C}_\varrho^\pm$, such that the term involving $B$ will be absent in (53) and we recover the result of [4]. The weak limit $B \to 0$ turns equ. (53) into a trivial identity by noting that with (48) $\lim_{B\to 0} \widetilde{\varphi}_{ij}(k) = 2\pi\delta_{ij}$.

As the last quantity which will be important for our considerations we have to compute the power series expansion of $\widetilde{\varphi}_{ij}(k)$. For this purpose we make use of the identity

$$\frac{\sinh(xt)}{\sinh t} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+x}{2}\right) t^{2n}\,, \tag{54}$$

which follows directly from the generating function for the Bernoulli polynomials $B_n(x)$ of degree $n$ (e.g. [28]). We may then expand each building block of the Fourier transformed TBA-kernel (45) as

$$\begin{aligned} \widetilde{\omega}_x(k) &= 2\pi \sum_{n=0}^{\infty} \mu_x^{(2n)} k^{2n} && (55)\\ &= 2\pi \sum_{n=0}^{\infty} \left(\nu_{x+1}^{(2n)} + \nu_{x-1}^{(2n)} - \nu_{x+1-B}^{(2n)} - \nu_{x-1+B}^{(2n)}\right) k^{2n} && (56) \end{aligned}$$

where the coefficients are $\nu_x^{(2n)} = 2(2\pi)^{2n}/(2n+1)! B_{2n+1}(x/(2h))$. Therefore we obtain for the Fourier coefficients in equation (5)

$$\eta_{ij}^{(0)} = \delta_{ij} - g_{ij} \quad \text{and} \quad \eta_{ij}^{(2n)} = \frac{(-1)^{n+1}}{2} \sum_{q=1}^{h} (\lambda_i \cdot \sigma^{q+(c(j)-1)/2} \gamma_j) \mu^{(2n)}_{2q+(c(j)-c(i))/2-1} \quad (57)$$

for $n = 1, 2, \ldots$ The second order coefficient is particularly important with regard to our approximated analytical solution presented in section 3.2 and we will therefore analyze it a bit further. From (56) we obtain $\mu_x^{(2)} = \pi^2 B(B-2)(h-x)/h^3$, such that

$$\eta_{ij}^{(2)} = \frac{1}{2h^3} \sum_{q=1}^{h} (\lambda_i \cdot \sigma^{q+(c(j)-1)/2} \gamma_j) \pi^2 B(B-2)(h - 2q - (c(j) - c(i))/2 + 1) \ . \quad (58)$$

We may evaluate the sum over $q$ by noting further that

$$\lambda_i = \frac{1}{h} \sum_{q=1}^{h} q \sigma^{q+(c(i)-1)/2} \gamma_i \qquad \text{and} \qquad \sum_{q=1}^{h} \lambda_i \cdot \sigma^q \gamma_j = 0 \ . \quad (59)$$

The first identity follows by inverting the relation $\gamma_i = (1 - \sigma^{-1}) \sigma^{(1-c(i))/2} \lambda_i$ [23] and the second by computing the geometric series. Therefore (58) becomes

$$\eta_{ij}^{(2)} = \frac{\pi^2 B(2-B)}{h^2} \lambda_i \cdot \lambda_j = \frac{\pi^2 B(2-B)}{h^2} K_{ij}^{-1} \quad . \quad (60)$$

The latter formula follows immediately by recalling that $\lambda_i = \sum_{j=1}^r K_{ij}^{-1} \alpha_j$. It has the virtue that it is directly applicable since it involves only quantities which may be effortlessly extracted from Lie algebraic tables. At last we may compute the quantity $\eta_i^{(2)} = \sum_{j=1}^r \eta_{ij}^{(2)}$ which occurs in our approximated analytical approach (20). Obviously we obtain from (60)

$$\eta_i^{(2)} = \frac{\pi^2 B(2-B)}{h^2} \lambda_i \cdot \rho \ , \quad (61)$$

where $\rho = \sum_{i=1}^r \lambda_i = 1/2 \sum_{\alpha > 0} \alpha$ is the Weyl vector. The inner product of $\lambda_i \cdot \rho$ may be related to a universal quantity, namely the index of the fundamental representation $\lambda_i$

$$x_{\lambda_i} = \frac{\dim \lambda_i}{2 \dim \mathbf{g}} (2\rho + \lambda_i) \cdot \lambda_i \ , \quad (62)$$

such that we finally obtain

$$\eta_i^{(2)} = \frac{\pi^2 B(2-B)}{2h^2} \left( \frac{2 x_{\lambda_i} \dim \mathbf{g}}{\dim \lambda_i} - K_{ii}^{-1} \right) \ . \quad (63)$$

Hence we are also able to evaluate $\eta_i^{(2)}$ in a universal manner. We may proceed further and also compute

$$\sum_{i=1}^r \eta_i^{(2)} = \frac{\pi^2 B(2-B)}{h^2} \rho^2 = \frac{\pi^2 B(2-B) Q_\psi \dim \mathbf{g}}{24 h^2} = \frac{\pi^2 B(2-B) r(h+1)}{12h} . \quad (64)$$

Here we used the Freudenthal-deVries strange formula $\rho^2 = Q_\psi \mathrm{dim}\mathbf{g}/24$ (see e.g. [29]), the fact that for simply laced algebras the eigenvalue of the quadratic Casimir operator is $Q_\psi = 2h$ and that we have dim$\mathbf{g}$$= r(h+1)$ for the dimension of the Lie algebra $\mathbf{g}$.

### 4.2.2 Non-simply laced Lie algebras

For the non-simply laced cases we may proceed similarly, just now we are lacking the universality of the previous subsection. Using the generalized blocks (40) instead of (31) we derive

$$\omega_{x,y}(\theta) = -i\frac{d}{d\theta}\ln\,[x,y]_\theta = -\frac{1}{\pi}\int_0^\infty \frac{dt}{\sinh t}\; g_{x,y,B}(t)\cos\left(\frac{\theta t}{\pi}\right), \tag{65}$$

$$= \gamma_{x-yB-1}(\theta) + \gamma_{x+yB+1}(\theta) - \gamma_{x+yB+B-1}(\theta) - \gamma_{x-yB-B+1}(\theta)\,. \tag{66}$$

The Fourier transformed of $\omega_{x,y}(\theta)$ reads

$$\widetilde{\omega}_{x,y}(k) = -\frac{\pi\, g_{x,y,B}(\pi k)}{\sinh(\pi k)} \tag{67}$$

$$= 2\pi\sum_{n=0}^{\infty}\mu_{x,y}^{(2n)}k^{2n} \tag{68}$$

$$= 2\pi\sum_{n=0}^{\infty}\left(\nu_{x+1+yB}^{(2n)} + \nu_{x-1-yB}^{(2n)} - \nu_{x+1-B-yB}^{(2n)} - \nu_{x-1+B+yB}^{(2n)}\right)k^{2n}\,. \tag{69}$$

In particular

$$\mu_{x,y}^{(2)} = \frac{B\pi^2(B-2)(H-x)(1-2y)}{H^3} \tag{70}$$

is important for our purposes.

## 4.3 The TBA-equations

Having computed the TBA-kernels, it appears at first sight that, apart from a simple substitution, there is not much more to be said about the form of the TBA-equations. However, Zamolodchikov observed in [4] that once the Fourier transformed TBA-kernels admit a certain representation (in our situation this is (48)), the TBA-equations may be cast into a very universal form.

We will exploit now the universal features derived in the preceding section. In order to keep the notation simple, we commence by choosing the statistical interaction to be fermionic at first. The generalization to generic Haldane statistics is straightforward thereafter. First of all, we Fourier transform the TBA-equations in the variant (8)

$$2\pi\widetilde{\xi}_i(k,r) = \sum_{j=1}^{r}\widetilde{\varphi}_{ij}(k)\widetilde{L}_j(k,r)\,. \tag{71}$$

For the reason that the Fourier transforms of $rm_i \cosh\theta$ and $\varepsilon_i(\theta, r)$ do not exist separately, we introduced here the quantity $\xi_i(\theta, r) = rm_i \cosh\theta - \varepsilon_i(\theta, r)$. This should be kept in mind before transforming back to the original variables. Substituting now the general form of $\widetilde{\varphi}_{ij}(k)$ from (48) into equation (71) and taking the inverse Fourier transformation thereafter yields

$$\xi_i(\theta, r) = \sum_{j=1}^{r} \left( I_{ij}(\xi_j - L_j) * \Omega_h \right)(\theta, r) + (L_i * \Omega_{h,B})(\theta, r) \ . \tag{72}$$

Here we introduced the universal kernels

$$\Omega_h(\theta) = \frac{h}{2\cosh\frac{h}{2}\theta} \quad \text{and} \quad \Omega_{h,B}(\theta) = \frac{2h \sin\frac{\pi}{2}B \cosh\frac{h}{2}\theta}{\cosh h\theta - \cos\pi B} \ . \tag{73}$$

Neglecting the coupling constant dependent term involving $\Omega_{h,B}(\theta)$, i.e. taking the limit $B \to i\infty$, the identities (72) coincide precisely with the equations (7) in [4]. This is to be expected since the latter equations describes the system which dynamically interacts via the minimal part of (30). Note further that the strong-weak duality is still preserved, i.e. $\Omega_{h,B}(\theta) = \Omega_{h,2-B}(\theta)$.

We may now easily derive the generalized form of (72) for generic types of Haldane statistics. Recalling that for the derivation of (8) all the $L-$functions were parameterized by $\mathcal{L}_i(\theta, r) = \ln\left(1 + e^{-\varepsilon_i(\theta,r)}\right)$ and the statistical interaction by $g_{ij} = \delta_{ij} - g'_{ij}$, the Fourier transformation reads

$$\widetilde{\xi}_i(k, r) = \sum_{j=1}^{r} \left[ g'_{ij}\, \widetilde{\mathcal{L}}_j(k, r) + \widetilde{\varphi}_{ij}(k) \widetilde{\mathcal{L}}_j(k, r)/2\pi \right] \ . \tag{74}$$

In the same manner as for the fermionic case we derive

$$\xi_i(\theta, r) = \sum_{j=1}^{r} \left[ I_{ij}(\xi_j - \sum_{l=1}^{r} (\delta_{jl} + g'_{jl})\mathcal{L}_l) * \Omega_h + g'_{ij}\mathcal{L}_j \right](\theta, r) + (\mathcal{L}_i * \Omega_{h,B})(\theta, r) \ . \tag{75}$$

Of course we recover (72) from (75) for $g'_{ij} = 0$.

As a final consistency check we carry out the limit to the free theory, that is $B \to 0$. We have

$$\lim_{B\to 0} (\mathcal{L}_i * \Omega_{h,B})(\theta, r) = \lim_{B\to 0} \frac{1}{2} \int_{-\infty}^{\infty} d\theta' \frac{hB \cosh\frac{h}{2}\theta'}{\cosh h\theta' - 1 + \pi^2 B^2/2} \mathcal{L}_i(\theta - \theta', r) \tag{76}$$

$$= \int_{-\infty}^{\infty} dt \lim_{B\to 0} \frac{1}{\pi} \frac{B}{t^2 + B^2} \mathcal{L}_i\left(\theta - \frac{2}{h} \operatorname{arsinh}\frac{t\pi}{2}, r\right) \tag{77}$$

$$= \mathcal{L}_i(\theta, r) \ . \tag{78}$$

In the last equality we employed the well known representation for the delta function $\pi\delta(x) = \lim_{\varepsilon\to 0} \varepsilon/(\varepsilon^2 + x^2)$. This means that in the extreme weak limit the TBA-equations (75) is solved by

$$\varepsilon_i(\theta, r) = rm_i \cosh\theta - \sum_{j=1}^{r} (\delta_{ij} + g'_{ij}) L_j(\theta, r) \ . \tag{79}$$

That (79) are indeed *the* TBA-equations may also be seen by taking this limes directly in (8) and noting that $\lim_{B\to 0} \varphi_{ij}(\theta) = \delta_{ij}(\theta)$. In this way we have "switched off" the dynamical interaction and obtained a system which interacts purely by statistics. In particular for fermionic type of statistics ($g'_{ij} = 0$) or for $g'_{ij} = -\delta_{ij}$ these equations describe $r$ free bosons or $r$ free fermions, respectively. Taking $g'_{ij}$ to be diagonal, the equations (79) are equivalent to a system whose S-matrix corresponds to the one of the Calogero-Sutherland model and whose statistical interaction is taken to be of fermionic type [18].

We finish this subsection by noting that $\sum_j I_{ij} m_j = 2m_i \cos(\pi/h)$ [30], from which follows

$$rm_i \cosh\theta = r \sum_{j=1}^{r} I_{ij} m_j \left(\cosh * \Omega_h\right)(\theta, r) \ , \tag{80}$$

such that the entire $r$-dependence and the mass spectrum in (72) and (75) may be eliminated. We obtain

$$\varepsilon_i(\theta, r) = \sum_{j=1}^{r} \left[ I_{ij}(\varepsilon_j + \sum_{l=1}^{r} (\delta_{jl} + g'_{jl}) \mathcal{L}_l) * \Omega_h - g'_{ij} \mathcal{L}_j \right](\theta, r) - (\mathcal{L}_i * \Omega_{h,B})(\theta, r) \tag{81}$$

instead. One may solve these equations and re-introduce the $r$-dependence thereafter by means of the asymptotic behaviour. This is similar to the situation in the next subsection, in which the equations are even further simplified to a set of functional equations.

## 4.4 Y-systems

An alternative analytical approach is provided by exploiting properties of what is often referred to as Y-systems [4]. In order to derive these equations, we invoke once more the convolution theorem on the convolution in equation (8) and compute thereafter the sum of (8) at $\theta + i\pi/h$ and $\theta - i\pi/h$ minus $\sum_j I_{ij}$ times (8).Using once more the fact that $\sum_j I_{ij} m_j = 2m_i \cos(\pi/h)$, the terms involving $m_i$ cancel. Employing thereafter the identity (53) and transforming back into the original $\theta-$space we obtain, upon the introduction of the new variable $Y_i(\theta) = \exp(-\varepsilon_i(\theta))$,

$$Y_i\left(\theta + \tfrac{i\pi}{h}\right) Y_i\left(\theta - \tfrac{i\pi}{h}\right) = \left[1 + Y_i\left(\theta + \tfrac{i\pi}{h}(1-B)\right)\right]\left[1 + Y_i\left(\theta - \tfrac{i\pi}{h}(1-B)\right)\right]$$
$$\times \prod_{j=1}^{r} \left(1 + Y_j^{-1}(\theta)\right)^{-I_{ji}} \left( \left[1 + Y_j\left(\theta + \tfrac{i\pi}{h}\right)\right]\left[1 + Y_j\left(\theta - \tfrac{i\pi}{h}\right)\right] \prod_{l=1}^{r} \left[1 + Y_l(\theta)\right]^{-I_{jl}} \right)^{g'_{ij}} .$$

For fermionic statistics, i.e. $g'_{ij} = 0$ we recover the systems stated in [12] ($h = 2$) and [13], when we chose the "roaming parameter" therein such that $S(\theta)$ equals (30). The strong-weak duality is evidently still preserved. The Y-systems corresponding to the "minimal theories" are obtained by taking the limit $B \to i\infty$, by noting that $\lim_{\theta\to\infty} Y_i(\theta) = 0$. The latter limes follows from the TBA-equations together with the explicit form of the TBA-kernel, such that for large $\theta$ we have $\varepsilon_i(\theta) \sim rm_i \cosh\theta$. For fermionic type of statistics the functional equations obtained in this limit coincide with those found in [4]. As is to be expected from (79), we obtain for $g'_{ij} = 0$ (for $h = 0$ this holds in general) in the extremely weak coupling limit

$$Y_i(\theta+2\pi i) = Y_i(\theta), \qquad B \to 0, 2 \ . \tag{82}$$

To derive these periodicities directly from the functional equations is rather cumbersome once the algebras are more complicated and we do not have a general case independent proof. This is similar to the situation in which the system involved is the minimal part of (30) [4]. We leave it for future investigations to exploit the functional equations further.

## 4.5 Scaling functions

As already stated, solving the TBA-equations and the subsequent evaluation of (4) leads to expressions of the scaling function $c(r)$ with the inverse temperature $r$ times a mass scale as the scaling parameter. We may carry out this task numerically. Alternatively, assembling our results from section 3 and the analysis of the TBA-kernel, we can write down a universal formula for the scaling function of the system in which the statistical interaction is of fermionic type and in which the dynamical interaction is governed by the scattering matrix of affine Toda field theories related to simply laced Lie algebras, up to the first leading order in the ultraviolet regime. From (29) and (64) it follows under the assumption that the constants $\beta_i$ are the same for all particle types

$$c^{\mathbf{g}}(r) = \mathrm{rank}\mathbf{g}\left(1 - \frac{\pi^2 B(2-B)(h+1)}{4h(\beta - \ln(r/2))^2}\right) \quad . \tag{83}$$

We will now compute step by step several scaling functions for affine Toda field theories in which we specify some concrete algebras. We gradually increase the complexity of the models and focus on different aspects.

### 4.5.1 $A_1^{(1)}$-ATFT (Sinh-Gordon)

The Sinh-Gordon theory is one of the simplest quantum field theories in 1+1 dimensions and therefore the ideal starting point to concretely test general statements made in this context. Semi-classical investigations of a TBA-type nature concerning this model may already be found in [31]. The model contains only one stable particle which does not fuse and its scattering matrix simply reads

$$S^{SG}(\theta) = \{1\}_\theta = \frac{\tanh\frac{1}{2}\left(\theta - \frac{i\pi}{2}B\right)}{\tanh\frac{1}{2}\left(\theta + \frac{i\pi}{2}B\right)} \quad . \tag{84}$$

For a system whose dynamics is described by this S-matrix and whose statistical interaction is of general Haldane type, the TBA-equation (2) reads

$$rm\cosh\theta + \ln(1 - e^{-L(\theta,r)}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' \varphi^{SG}(\theta')\, L(\theta - \theta', r) - gL(\theta, r) \tag{85}$$

with

$$\varphi^{SG}(\theta) = \Omega_{2,B}(\theta) = \frac{4\sin(\pi B/2)\cosh\theta}{\cosh 2\theta - \cos\pi B} . \tag{86}$$

We may convince ourselves that (85) can be obtained from (2) and the direct computation of the kernel as well as from (75) by noting that the incidence matrix of course vanishes for the $A_1^{(1)}$ related Dynkin diagram.

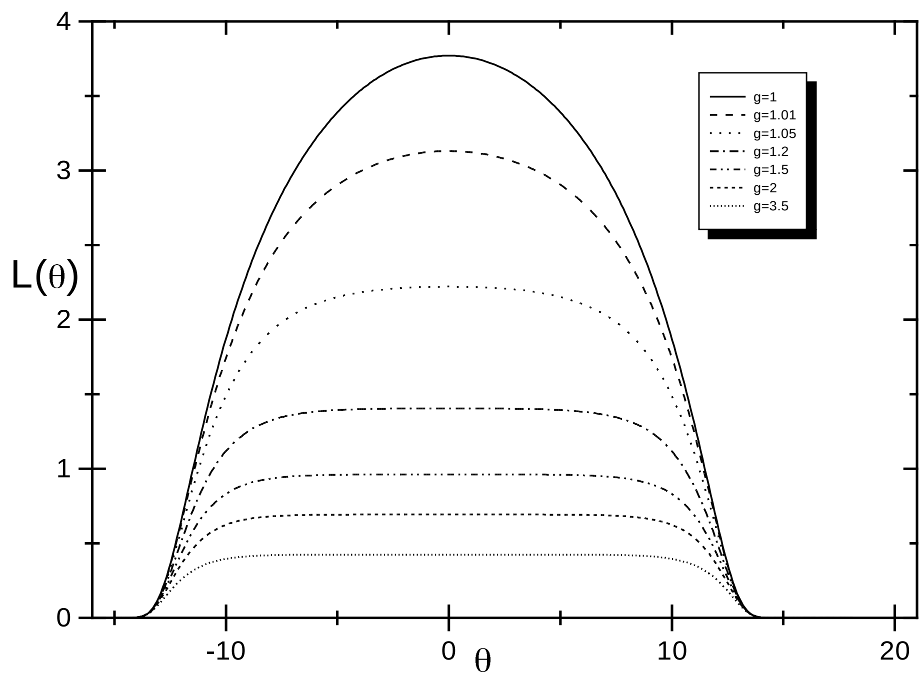


Figure 2: Numerical solution for the Sinh-Gordon related TBA-equations for different statistical interactions $g$ and fixed values of the effective coupling $B = 0.4$ and $r = 10^{-5}$.

The weak coupling limit $B \to 0$ is read off from (79)

$$rm\cosh\theta + \ln(1 - e^{-L(\theta,r)}) + gL(\theta, r) = L(\theta, r) \quad . \tag{87}$$

The other limes which may be computed in the standard way is the limes $r \to 0$, leading to the constant TBA-equation

$$\ln\left(1 - e^{-L(0,0)}\right) = (1-g)L(0,0) \quad . \tag{88}$$

This is obviously compatible with equation (87) and indicates that the extreme ultraviolet behaviour is independent of the dynamics and is purely governed by the statistics. In particular we obtain for fermionic type of statistics $L(0,0) \to \infty$ as a solution of (88), such that with (18) the effective central charge turns out to be one. In general (88) yields $0 \leq c_{\text{eff}} \leq 1$ for $g \geq 1$. We also observe that due to the fact that $L(0,0) \geq 0$, equation (88) does not possess any physical solutions at all for $g < 1$, which means in particular that there are no solutions for the statistical interaction of bosonic type.

We shall now turn to the full solution of the TBA-equation (85). First of all we solve this equation numerically. For $g \geq 1$ we observe in figure 2 the typical behaviour, known from the analysis of the minimal part of the S-matrices [3], that the $L$-function is essentially constant between $\pm 2\ln(r/2)$. As expected from the previous discussion, also the numerical analysis does not yield any solution for $g < 1$. For a more detailed discussion concerning the nature of the numerical procedure and in particular the question of convergence and uniqueness of the solutions we refer to section 5.

We shall now compare our numerical results with our analytic approximations of section 3. The general formulae (45) or (48) yields for the Fourier transform of the TBA-kernel

$$\widetilde{\Phi}^{SG}(k) = \frac{2\pi \cosh\left(\frac{k\pi}{2} - \frac{Bk\pi}{2}\right)}{\cosh\left(\frac{k\pi}{2}\right)} - 2\pi g \quad . \tag{89}$$

For the reasons mentioned in section 3.2, we will now restrict ourselves to fermionic type of statistics, such that

$$\eta^{(0)} = 0 \qquad \text{and} \qquad \eta^{(2)} = \frac{\pi^2 B(2-B)}{8} \, . \tag{90}$$

We obtain a fairly good agreement between the numerical solution of (2) for the function $L(\theta, r)$ and the approximated analytical solution (24), if we choose the constants in (20) to be

$$\beta = \ln(B(2-B)2^{1+B(2-B)}) \qquad \text{and} \qquad \alpha = \frac{\pi}{2(\beta - \ln(r/2))} \;, \tag{91}$$

such that

$$L(\theta, r) = \ln\left(\cos^2(\alpha\theta)\right) - \ln[B(2-B)(\pi\alpha/2)^2] \; . \tag{92}$$

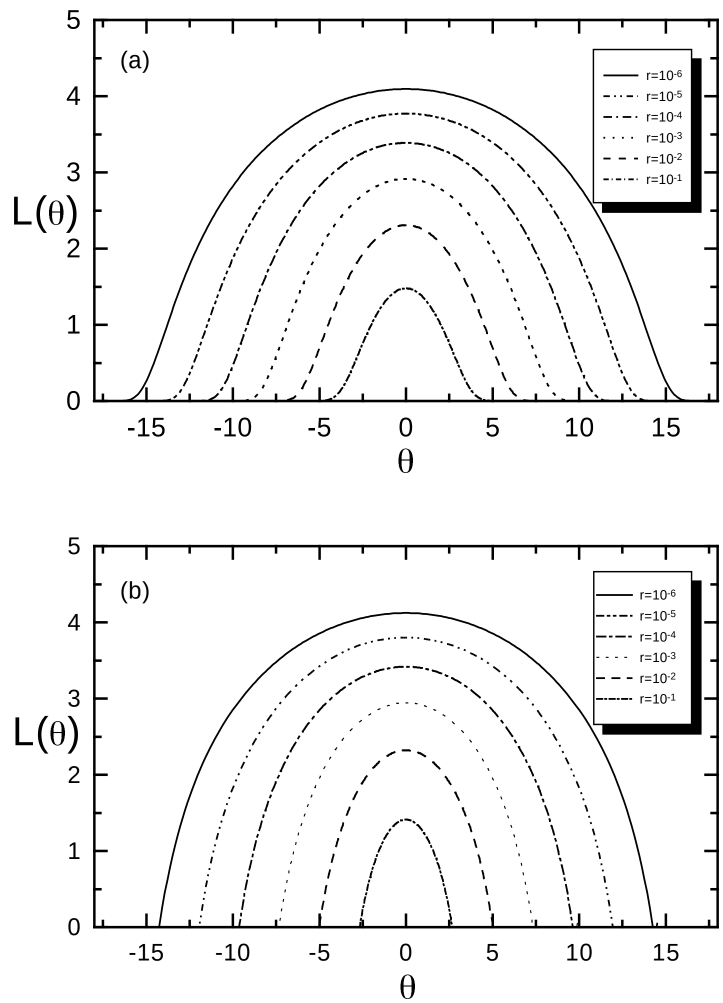


Figure 3: Numerical solution (a) versus approximated analytical solution (b) for the Sinh-Gordon related TBA-equation for various values of $r$ and fixed effective coupling $B = 0.4$.

Figure 3 shows that the $r$ dependence of $L(\theta, r)$ is captured very well by the approximated analytical solution (92). As expected from the nature of the assumptions made in the derivation, (24) becomes relatively poor when $L(\theta, r)$ is small. With regard to the aim of these computations, that is the evaluation of the scaling function $c(r)$, precisely in this region the error is negligible as is seen from (4). Notice also that the assumptions (16) are justified in retrospective by the numerical solutions. The quality of the approximation may also be seen by comparing tables 1-3 for large values of $L(\theta, r)$.

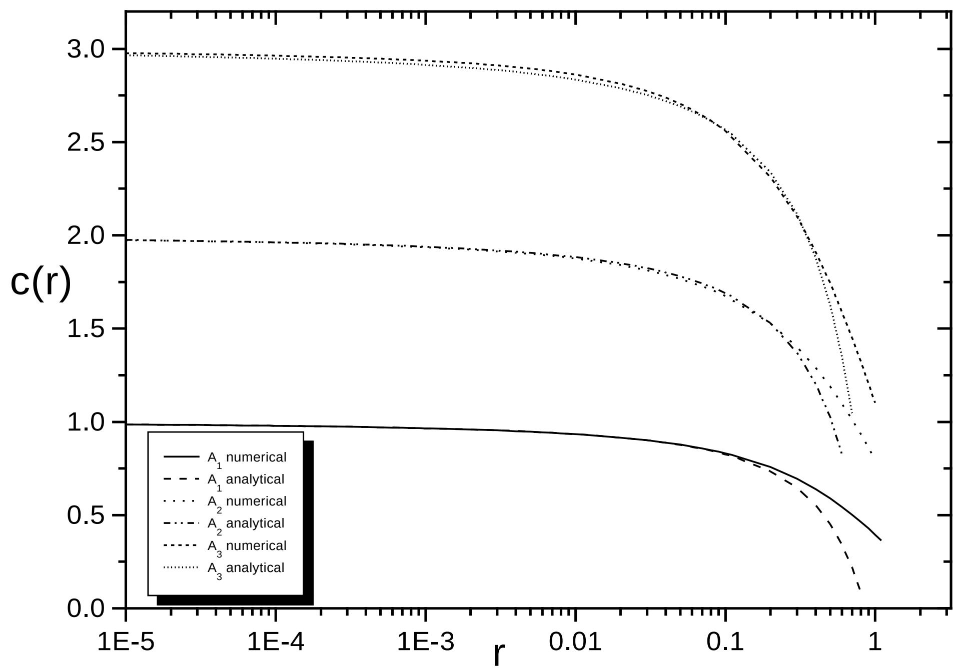


Figure 4: Numerical solution versus approximated analytical solution for the scaling function of the $A_1^{(1)}, A_2^{(1)}$ and $A_3^{(1)}$-affine Toda field theory related TBA-systems with fixed effective coupling $B = 0.4$.

Finally we compute the scaling function, which acquires in this approximation the form

$$c^{SG}(r) = 1 - \frac{3\,\pi^2 B(2-B)}{8(\beta - \ln(r/2))^2} \quad . \tag{93}$$

Also this function is well approximated in the ultraviolet regime ($r < 1$) as is seen from figure 4 and the tables 1-3.

| $r$ | $L(0,r)$/n. | $L(0,r)$/a. | $c(r)$/ n. | $c(r)$/a. |
|---|---|---|---|---|
| $10^{-1}$ | 1.845 | 2.135 | 0.8767 | 0.9281 |
| $10^{-2}$ | 2.957 | 3.238 | 0.9648 | 0.9762 |
| $10^{-3}$ | 3.722 | 3.945 | 0.9845 | 0.9882 |
| $10^{-4}$ | 4.286 | 4.467 | 0.9914 | 0.9930 |
| $10^{-5}$ | 4.740 | 4.880 | 0.9949 | 0.9954 |
| $10^{-6}$ | 5.153 | 5.222 | 0.9975 | 0.9967 |

Table 1: Numerical solution (n.) versus approximated analytical solution (a.) for the Sinh-Gordon related TBA-system with fixed effective coupling $B = 0.1$.

| $r$ | $L(0)$/n. | $L(0)$/a. | $c(r)$/ n. | $c(r)$/a. |
|---|---|---|---|---|
| $10^{-1}$ | 1.478 | 1.249 | 0.8311 | 0.8257 |
| $10^{-2}$ | 2.307 | 2.220 | 0.9339 | 0.9340 |
| $10^{-3}$ | 2.915 | 2.870 | 0.9654 | 0.9655 |
| $10^{-4}$ | 3.387 | 3.361 | 0.9789 | 0.9789 |
| $10^{-5}$ | 3.771 | 3.754 | 0.9857 | 0.9858 |
| $10^{-6}$ | 4.094 | 4.082 | 0.9897 | 0.9897 |

Table 2: Numerical solution (n.) versus approximated analytical solution (a.) for the Sinh-Gordon related TBA-system with fixed effective coupling $B = 0.4$.

| $r$ | $L(0,r)$/n. | $L(0,r)$/a. | $c(r)$/ n. | $c(r)$/a. |
|---|---|---|---|---|
| $10^{-1}$ | 1.335 | 0.811 | 0.8070 | 0.7297 |
| $10^{-2}$ | 2.074 | 1.782 | 0.9165 | 0.8977 |
| $10^{-3}$ | 2.631 | 2.433 | 0.9540 | 0.9466 |
| $10^{-4}$ | 3.072 | 2.923 | 0.9709 | 0.9673 |
| $10^{-5}$ | 3.435 | 3.317 | 0.9800 | 0.9780 |
| $10^{-6}$ | 3.743 | 3.646 | 0.9853 | 0.9841 |

Table 3: Numerical solution (n.) versus approximated analytical solution (a.) for the Sinh-Gordon related TBA-system with fixed effective coupling $B = 0.9$.

#### 4.5.2 $A_2^{(1)}$-ATFT/$A_2^{(2)}$-ATFT (Bullough-Dodd)

In comparison to the Sinh-Gordon model the next complication arises when we allow the particle in the system to fuse. This case will arise when we consider the $A_2^{(1)}$- and $A_2^{(2)}$-ATFT, which are known to be closely related. The former contains two particles which are conjugate to each other, whereas the latter contains only one particle which is the bound state of itself. With regard to the TBA-analysis these models may be treated on the same footing. The corresponding two particle scattering matrices read

$$S_{11}^{A_2^{(1)}}(\theta) = S_{22}^{A_2^{(1)}}(\theta) = \{1\}_\theta \qquad S_{12}^{A_2^{(1)}}(\theta) = \{2\}_\theta \qquad S^{A_2^{(2)}}(\theta) = \{1\}_\theta \{2\}_\theta \,. \tag{94}$$

Due to the fact that $S^{A_2^{(2)}}(\theta) = S_{11}^{A_2^{(1)}}(\theta)\, S_{12}^{A_2^{(1)}}(\theta)$, the TBA-equation of the $A_2^{(2)}$-theory equals the two TBA-equations of the $A_2^{(1)}$-theory under the natural assumptions that $L_1(\theta) = L_2(\theta)$ and $g_{ij} = g_{ji}$. The common TBA-kernel without the statistics factor is computed to

$$\varphi^{BD}(\theta) = \frac{-4\sqrt{3}\cosh\theta}{2\cosh(2\theta)+1} + \frac{4\cosh\theta\sin((1+B)\pi/3)}{\cosh(2\theta)-\cos((1+B)2\pi/3)} + \frac{4\cosh\theta\sin(B\pi/3)}{\cosh(2\theta)-\cos(2B\pi/3)}\,. \tag{95}$$

We may then write the TBA-equation either in its variant (2) as

$$rm\cosh\theta + \ln(1 - e^{-L(\theta,r)}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' \varphi^{BD}(\theta')\, L(\theta - \theta', r) - gL(\theta, r) \qquad (96)$$

or in its universal form (75) as

$$\xi(\theta, r) = \left[(\xi - (1 + g')\mathcal{L}) * \Omega_3\right](\theta, r) + g'\mathcal{L}(\theta, r) + (\mathcal{L}_i * \Omega_{h,B})(\theta, r)\,. \qquad (97)$$

We used the abbreviations $g = g_{11} + g_{12}$ and $g' = g'_{11} + g'_{12}$. The Fourier transformed TBA-kernel is computed from (45) or (48)

$$\widetilde{\Phi}^{A_2^{(2)}}(k) = \widetilde{\Phi}_{11}^{A_2^{(1)}}(k) + \widetilde{\Phi}_{12}^{A_2^{(1)}}(k) = \widetilde{\Phi}_{21}^{A_2^{(1)}}(k) + \widetilde{\Phi}_{22}^{A_2^{(1)}}(k) \qquad (98)$$

$$= \frac{4\pi\left(\sinh\left(\frac{\pi k}{3}\right) + \sinh\left(\frac{2\pi k}{3}\right)\right)\left(\cosh\left(\frac{k\pi}{3} - \frac{Bk\pi}{3}\right) - \frac{1}{2}\right)}{\sinh(\pi k)} - 2\pi g\,. \qquad (99)$$

We restrict now to fermionic type of statistics and chose the constants in our analytic solution (20) in the same way as for the Sinh-Gordon model, i.e.

$$\beta = \beta_1 = \beta_2 = \ln(B(2-B)2^{1+B(2-B)}) \quad \text{and} \quad \alpha = \alpha_1 = \alpha_2 = \frac{\pi}{2(\beta - \ln(r/2))}\,, \qquad (100)$$

such that the approximated function for $L(\theta, r)$ equals (92). The scaling function up to the first leading order (29) becomes now

$$c^{A_2^{(1)}}(r) = 2c^{A_2^{(2)}}(r) = 2 - \frac{2\,\pi^2 B(2-B)}{3(\beta - \ln(r/2))^2}\,. \qquad (101)$$

Solving (96) numerically we obtain qualitatively the same kind of agreement with the approximated analytical solution as for the Sinh-Gordon model, see figure 4.

### 4.5.3 $A_3^{(1)}$-ATFT

The simplest model which involves two inequivalent TBA-equations coupled to each other is the $A_3^{(1)}$-affine Toda field theory. In our conventions the two-particle scattering matrices for this theory read

$$S_{11}^{A_3^{(1)}}(\theta) = S_{33}^{A_3^{(1)}}(\theta) = \{1\}_\theta\,, \qquad S_{13}^{A_3^{(1)}}(\theta) = \{3\}_\theta\,, \qquad (102)$$

$$S_{12}^{A_3^{(1)}}(\theta) = S_{23}^{A_3^{(1)}}(\theta) = \{2\}_\theta\,, \qquad S_{22}^{A_3^{(1)}}(\theta) = \{1\}_\theta\,\{3\}_\theta\,. \qquad (103)$$

Particle 1 is the anti-particle of 3 and particle 2 is self-conjugate. The masses are $m_1 = m_3 = m/\sqrt{2}$ and $m_2 = m$. In the direct channel the fusings $11 \to 2$ and

$33 \to 2$ are possible. Under the assumption that $\varepsilon_1(\theta,r) = \varepsilon_3(\theta,r)$, the TBA-equations in the variant (8) now read

$$\begin{aligned}
\varepsilon_1(\theta,r) &= rm/\sqrt{2}\cosh\theta - \left(\varphi_{22}^{A_3^{(2)}} * \mathcal{L}_1 + \varphi_{12}^{A_3^{(2)}} * \mathcal{L}_2\right)(\theta,r) \\
&\quad -(g'_{11}+g'_{13})\mathcal{L}_1(\theta,r) - g'_{12}\mathcal{L}_2(\theta,r), \qquad (104)\\
\varepsilon_2(\theta,r) &= rm\cosh\theta - \left(\varphi_{22}^{A_3^{(2)}} * \mathcal{L}_2 + 2\varphi_{12}^{A_3^{(2)}} * \mathcal{L}_1\right)(\theta,r) \\
&\quad -(g'_{12}+g'_{23})\mathcal{L}_1(\theta,r) - g'_{22}\mathcal{L}_2(\theta,r)\,, \qquad (105)
\end{aligned}$$

where the kernels are

$$\begin{aligned}
\varphi_{22}^{A_3^{(2)}}(\theta) &= \frac{\sin(B\pi/4)}{\cosh^2\theta - \cos^2(B\pi/4)} + \frac{\cos(B\pi/4)}{\cosh^2\theta + \sin^2(B\pi/4)} - \frac{2}{\cosh\theta} \qquad (106)\\
\varphi_{12}^{A_3^{(2)}}(\theta) &= 2\cosh\theta\left(\frac{2\sin((B+1)\pi/4)}{\cosh(2\theta)+\sin(B\pi/2)} - \frac{\sqrt{2}}{\cosh(2\theta)}\right). \qquad (107)
\end{aligned}$$

Note that $\varphi_{12}^{A_3^{(2)}}(\theta) = \varphi_{23}^{A_3^{(2)}}(\theta)$ and $\varphi_{22}^{A_3^{(2)}}(\theta) = \varphi_{11}^{A_3^{(2)}}(\theta) + \varphi_{13}^{A_3^{(2)}}(\theta)$. Alternatively we may write down the universal form of the TBA-equations (72), which becomes particularly simple for the fermionic statistics

$$\begin{aligned}
\xi_1(\theta,r) &= (\xi_2 - L_2) * \Omega_4(\theta,r) + (L_1 * \Omega_{4,B})(\theta,r) \qquad (108)\\
\xi_2(\theta,r) &= 2(\xi_1 - L_1) * \Omega_4(\theta,r) + (L_2 * \Omega_{4,B})(\theta,r)\,. \qquad (109)
\end{aligned}$$

We compute from (45) or (48) the Fourier transformed TBA-kernels

$$\begin{aligned}
\widetilde{\varphi}_{22}^{A_3^{(2)}}(k) &= 2\pi\frac{(1+2\cosh(\pi k/4))(\cosh((B-1)\pi k/4) - \cosh\pi k/4)}{\cosh(k\pi/4)+\cosh(3k\pi/4)} \qquad (110)\\
\widetilde{\varphi}_{12}^{A_3^{(2)}}(k) &= 2\pi\frac{\cosh(k\pi/4)\cosh((B-1)k\pi/4) - 1}{\cosh(k\pi/2)}\,. \qquad (111)
\end{aligned}$$

From (60) or from the explicit expression for $\mu_x^{(2)}$ corresponding to each block in (102) and (103), we obtain for fermionic statistics

$$\eta_{ij}^{(0)} = 0 \quad \text{and} \quad \eta_{ij}^{(2)} = \frac{\pi^2 B(2-B)}{4^3}\begin{pmatrix} 3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{pmatrix}_{ij}\,. \qquad (112)$$

Therefore we have

$$\eta_1^{(2)} = \eta_3^{(2)} = \frac{3\pi^2 B(2-B)}{32} \quad \text{and} \quad \eta_2^{(2)} = \frac{\pi^2 B(2-B)}{8}\,. \qquad (113)$$

Computing the sum $\eta_1^{(2)}+\eta_2^{(2)}+\eta_3^{(2)}$ confirms our general formula (64). Finally we obtain for the approximated scaling function

$$c(r)=3-\frac{15\pi^2 B(2-B)}{16(\beta-\ln(r/2))^2}\ . \tag{114}$$

Once more we compare this analytical expression with the numerical solution of the two coupled TBA-equations (104) and (105). Choosing the constants in the same way as for the Sinh-Gordon model, figure 4 demonstrates the same kind of qualitative agreement as we observed for the previous models. For all models we observe that at about $r=0.1$ the analytical expressions for $c(r)$ start to decrease more rapidly than the numerical solution.

#### 4.5.4 $(G_2^{(1)} \Leftrightarrow D_4^{(3)})$-ATFT

This theory is the simplest example for a model in which the masses as a function of the coupling constant flow from the classical mass spectrum of one Lagrangian to its dual in the Lie algebraic sense. The theory contains two self-conjugate particles whose masses were conjectured [32, 33] (supported by numerical investigations [34]) to $m_1=m\sin(\pi/H)$ and $m_2=m\sin(2\pi/H)$. The "floating Coxeter number" is taken to be $H=6+3B$ with $0\leq B\leq 2$. The fusing processes $1\,1\rightarrow 1+2$, $2\,2\rightarrow 1+2+2$ and $1\,2\rightarrow 1+2$ are possible. In our conventions the related scattering matrices [32, 26] read

$$S_{11}^{G/D}(\theta)=[1,0]_\theta[H-1,0]_\theta[H/2,1/2]_\theta,\quad S_{12}^{G/D}(\theta)=[H/3,1]_\theta[2H/3,1]_\theta, \tag{115}$$
$$S_{22}^{G/D}(\theta)=[H/3-1,1]_\theta[H/3+1,1]_\theta[2H/3-1,1]_\theta[2H/3+1,1]_\theta\quad . \tag{116}$$

Besides the loss of duality these scattering matrices exhibit a further difference in comparison with the simply laced case, which has a bearing on the TBA-analysis. In the standard prescription the symmetry of the Bethe wave function is derived by exploiting the behaviour of the scattering matrices at $\theta=0$. As already mentioned for the simply laced case we always have $S_{ij}(0)=(-1)^{\delta_{ij}}$, whereas now we observe $S_{22}^{G/D}(0)=-S_{11}^{G/D}(0)=1$. Assuming that the particles described are bosons, we should choose, according to the arguments of [2], $g_{11}=1$ and $g_{22}=0$. In this case, however, the TBA-analysis does not produce any physical solution. The TBA-equations in the variant (8) become

$$\begin{aligned}\varepsilon_1(\theta,r) &= rm\sin(\pi/H)\cosh\theta-\left(\varphi_{11}^{G/D}*\mathcal{L}_1\right)(\theta,r)-\left(\varphi_{12}^{G/D}*\mathcal{L}_2\right)(\theta,r)\\ &\quad -g'_{11}\mathcal{L}_1(\theta,r)-g'_{12}\mathcal{L}_2(\theta,r)\ , \end{aligned}\tag{117}$$
$$\begin{aligned}\varepsilon_2(\theta,r) &= rm\sin(2\pi/H)\cosh\theta-\left(\varphi_{21}^{G/D}*\mathcal{L}_1\right)(\theta,r)-\left(\varphi_{22}^{G/D}*\mathcal{L}_2\right)(\theta,r)\\ &\quad -g'_{21}\mathcal{L}_1(\theta,r)-g'_{22}\mathcal{L}_2(\theta,r)\ . \end{aligned}\tag{118}$$

The kernels and their Fourier transformed versions read

$$\varphi_{11}^{G/D}(\theta) = \omega_1(\theta) + \omega_{H-1}(\theta) + \omega_{H/2,1/2}(\theta) \tag{119}$$
$$\varphi_{12}^{G/D}(\theta) = \varphi_{21}(\theta) = \omega_{H/3,1}(\theta) + \omega_{2H/3,1}(\theta) \tag{120}$$
$$\varphi_{22}^{G/D}(\theta) = \omega_{H/3-1,1}(\theta) + \omega_{H/3+1,1}(\theta) + \omega_{2H/3-1,1}(\theta) + \omega_{2H/3+1,1}(\theta) \tag{121}$$

and

$$\widetilde{\varphi}_{12}^{G/D}(k) = 4\pi\frac{\cosh((1+B)\pi k/H) - \cosh((1-2B)\pi k/H)}{1 - 2\cosh(\pi k/3)} \tag{122}$$
$$\widetilde{\varphi}_{11}^{G/D}(k) = \frac{\widetilde{\varphi}_{12}^{G/D}(k) + 4\pi\cosh((1-B)\pi k/H)}{2\cosh(\pi k/H)} \tag{123}$$
$$\widetilde{\varphi}_{22}^{G/D}(k) = 2\cosh(\pi k/H)\widetilde{\varphi}_{12}^{G/D}(k) + 2\pi\ . \tag{124}$$

From this or our general formulae of section 4.2. we obtain

$$\eta_{11}^{(2)} = \frac{\pi^2 B(2-B)}{H^2}\,, \quad \eta_{12}^{(2)} = -\eta_{11}^{(2)}\,, \quad \eta_{22}^{(2)} = -2\eta_{11}^{(2)}, \quad \eta_1^{(2)} = 0, \quad \eta_2^{(2)} = -3\eta_{11}^{(2)}\ . \tag{125}$$

The facts that $\eta_1^{(2)}$ vanishes and $\eta_2^{(2)}$ becomes negative make it impossible to use our approximated analytical solution (20). From a numerical point of view not much has changed in comparison with the previous models and we can still solve the related TBA-equations in the standard way. The related $L$-functions look qualitatively the same as in the previous cases and the scaling functions for some values of the effective coupling are depicted in figure 5 together with the same function for the $A_2^{(1)}$-affine Toda field theory. We observe that for fixed value of $r$, the scaling function for the $(G_2^{(1)}, D_4^{(3)})$-affine Toda field theory is a monotonically increasing function of $B$. Under the same circumstances the scaling functions related to theories in which the underlying algebras are simply laced are monotonically decreasing at first up to the self-dual point $B = 1$ and monotonically increase thereafter.

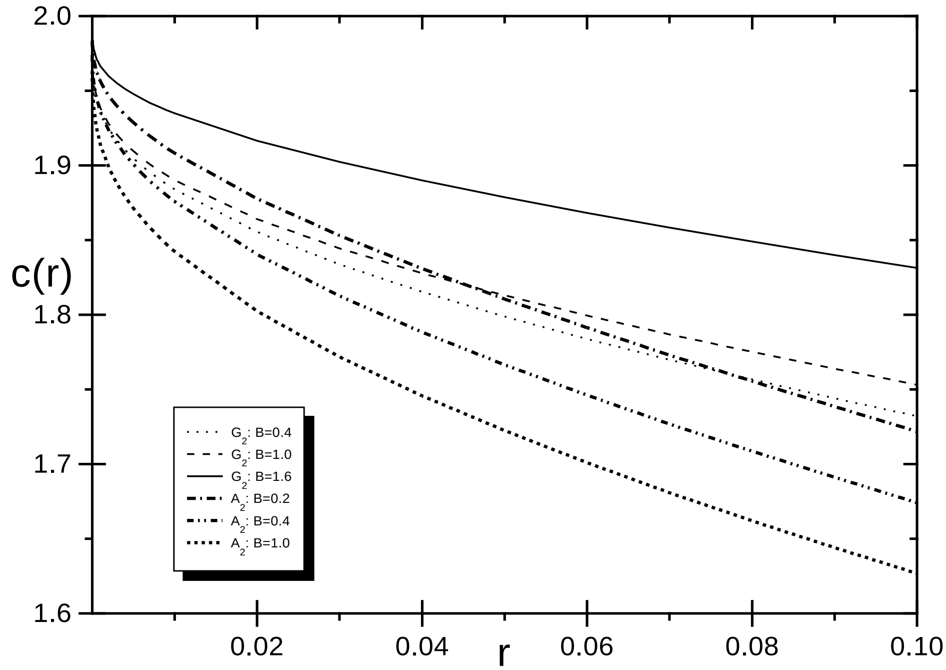


Figure 5: Scaling functions for the $A_2^{(1)}$ and $(G_2^{(1)}, D_4^{(3)})$-affine Toda field theory related TBA-systems for various values of the effective coupling.

# 5 Existence and Uniqueness Properties

In this section we are going to investigate the existence and uniqueness properties of the solutions of the TBA equations. Our main physical motivation for this considerations is to clarify whether it is possible to obtain different effective central charges for a fixed dynamical and statistical interaction due to the existence of several different solutions of the TBA equation. As a side product we obtain useful estimates on the error and the rate of convergence of the applied numerical procedure. Precise estimates of this kind were not obtained previously in this context and convergence is simply presumed. The method we employ is the contraction principle (or Banach fixed point theorem), see e.g. [35]. For the Yang–Yang equation in the case of the non–relativistic one–dimensional Bose gas for fermionic type of statistics the uniqueness question was already addressed by Yang and Yang [1], albeit with a different method.

In order to keep the notation simple we commence our discussion for a system with one particle only and a statistics of fermionic type. Thereafter we discuss the straightforward generalization. The standard way to solve integral equations of the type (8) consists in discretising the original function and a subsequent iteration, usually numerically. Normalizing the mass to one, this means we consider (8) as

$$\varepsilon_{n+1}(\theta, r) := r\cosh\theta - \varphi * \ln(1 + e^{-\varepsilon_n(\theta,r)}) \tag{126}$$

and perform the iteration starting with $\varepsilon_0(\theta, r) = r\cosh\theta$. The exact solution is then thought to be the limes $\lim_{n\to\infty}\varepsilon_n$. However, a priori it is not clear whether this limes exists at all and how it depends on the initial value $\varepsilon_0$. In particular, different initial values might lead to different solutions.

The natural mathematical setup for this type of problem is to re-write the TBA-equation as

$$(A\xi)(\theta, r) := \varphi * \ln(1 + e^{\xi(\theta,r) - r\cosh\theta}) = \xi(\theta, r), \tag{127}$$

and treat it as a fixed point problem for the operator $A$.

In order to give meaning to the limes $\lim_{n\to\infty}\varepsilon_n$ we have to specify a norm. Of course it is natural to assume that $\xi$ as function of $\theta$ is measurable, continuous and essentially bounded on the whole real line. The latter assumption is supported by all known numerical results. Furthermore it follows from (127), together with the explicit form of $\varphi$, that possible solutions $\xi$ vanish at infinity. This means possible solutions of (127) constitute a Banach space with respect to the norm

$$\|f\|_\infty = ess\sup|f(\theta)|\,, \tag{128}$$

i.e. $L_\infty(\mathbb{R})$. In principle we are now in a position to apply the Banach fixed point theorem[‖], which states the following:

Let $D \subset L_\infty$ be a nonempty set in a Banach space and let $A$ be an operator which maps $D$ $q$-contractively into itself, i.e. for all $f, g \in D$ and some fixed $q$, $0 \leq q < 1$

$$\|A(f) - A(g)\|_\infty \leq q\|f - g\|_\infty. \tag{129}$$

Then the following statements holds:

i) There exists a unique fixed point $\xi$ in $D$, i.e. equation (127) has exactly one solution.

ii) The sequence constructed in (126) by iteration converges to the solution of (127).

iii) The error of the iterative procedure may be estimated by

$$\|\xi - \xi_n\|_\infty \leq \frac{q^n}{1-q}\|\xi_1 - \xi_0\|_\infty \quad \text{and} \quad \left\|\xi - \xi_{n+1}\right\|_\infty \leq \frac{q}{1-q}\left\|\xi_{n+1} - \xi_n\right\|_\infty .$$

[‖] One may of course apply different types of fixed point theorems exploiting different properties of the operator $A$. For instance if $A$ is shown to be compact one can employ the Leray-Schauder fixed point theorem. In the second refence of [5] it is claimed that the problem at hand was treated in this manner, albeit a proof was not provided.

iv) The rate of convergence is determined by

$$\left\|\xi - \xi_{n+1}\right\|_\infty \leq q \left\|\xi - \xi_n\right\|_\infty .$$

In order to be able to apply the theorem we first have to choose a suitable set in the Banach space. We choose some $q \in [0,1)$ such that $e^{-r} \leq q$ and take $D$ to be the convex** set $D_{q,r} := \left\{ f : \; \|f\|_\infty \leq \ln \frac{q}{1-q} + r \right\}$. We may now apply the following estimate for the convolution operator $\varphi *$ (which is a special case of Young's inequality)

$$\|\varphi * f\|_\infty \leq \|\varphi\|_1 \|f\|_\infty . \tag{130}$$

For the concrete one particle models at hand, the Sinh-Gordon- and the Bullough-Dodd-model, we have $\|\varphi\|_1 := \int \frac{d\theta}{2\pi} |\varphi(\theta)| = 1$. For the operator $L$ we have the estimate

$$L(f) \leq \ln\left[1 + \exp\left(\|f\|_\infty - r\right)\right] \leq \ln \frac{1}{1-q} \leq \ln \frac{q}{1-q} + r.$$

The last inequality follows from our special choice of $q$. Thus, $A$ maps $D_{q,r}$ into itself.

In the final step we show that the contraction property (129) is fulfilled on $D_{q,r}$ . It suffices to prove this for the map $L$, because of (130) and the fact that $\|\varphi\|_1 = 1$. We have

$$\begin{aligned}
\|L(f) - L(g)\|_\infty &= \left\| \int_0^1 dt \, \frac{d}{dt} L(g + t(f-g)) \right\|_\infty \\
&= \left\| \int_0^1 dt \, \frac{(f-g)}{1 + \exp(-g - t(f-g) + r\cosh\theta)} \right\|_\infty \\
&\leq \max_{0 \leq t \leq 1} \left| \frac{1}{1 + \exp(-g - t(f-g) + r)} \right| \|f - g\|_\infty \\
&\leq \max_{0 \leq t \leq 1} \left| \frac{1}{1 + \exp\left(-\|g - t(f-g)\|_\infty + r\right)} \right| \|f - g\|_\infty \\
&\leq q \|f - g\|_\infty \; .
\end{aligned}$$

In the last inequality we used the fact that $D_{q,r}$ is a convex set.

We may now safely apply the fixed point theorem. First of all we conclude from i) and ii) that a solution of (127) not only exists, but it is also unique. In addition we can use iii)and iv) as a criterium for error estimates. From our special choice of the closed set $D_{q,r}$ one sees that the rate of convergence depends crucially on the parameter $r$, the smaller $r$ the greater $q$ is, whence the sequence $(\xi_n)$ converges slower.

**For $f, g \in D_{q,r}$ also $tf + (1-t)g \in D_{q,r}$, with $0 \leq t \leq 1$.

One could be mathematically more pedantic at this point and think about different requirements on the function $\xi$. For instance one might allow functions which are not bounded (we do not know of any example except when $r \to 0$) and then pursue similar arguments as before on $L_p$ rather than $L_\infty$.

The generalization of the presented arguments to a situation involving $l$ different types of particles and general Haldane statistics may be carried out in a straightforward manner.

# 6 Conclusions

Clearly an unsatisfactory feature of our approximated analytical expression for the scaling function (83) is that, at the moment, we do not have any additional constraint at hand which allows to determine the constant $\beta$. Admittedly the choice (91) enters our analysis in a rather ad hoc way. At the moment the rational behind our choice is that we may compare it with the semi-classical results in the spirit of [12] and thereafter restore the strong-weak duality. In addition it is supported by our numerical results. Surely this is by no means unique and it is highly desirable to eliminate this ambiguity.

Nonetheless, different choices of the constant will not change the overall behaviour and our expression is clearly in conflict with the results of Cassi and Destri [16], who found $f(r) = -3r^2 m_1^2/(\pi \varphi_{11}^{(1)})$ for the function in equation (1). The convention therein are that 1 labels the particle type with smallest mass and $\varphi_{11}^{(1)}$ results from the power series $\varphi_{11}(\theta) = -\sum_{k=1}^{\infty} \varphi_{11}^{(k)} \exp(-k\,|\theta|)$. On the side of the TBA-analysis the origin of this discrepancy may be tracked down easily. A behaviour of the type quoted in [16] was derived before in [12, 5] for affine Toda field theories related to the minimal part of the scattering matrix (30). This result was then extrapolated by Cassi and Destri in the way that $\varphi_{11}^{(1)}$ was taken to be the coefficient of the power series related to the full coupling constant dependent scattering matrix. However, the derivations in [12, 5] rely heavily on the assumption that the function $L(\theta)$ is constant in the region $-\ln(2/r) \ll \theta \ll \ln(2/r)$. It is essentially this property which is lost for the full theory (as our numerical results demonstrate), such that the arguments of Cassi and Destri become faulty. It would be interesting to settle the question also on the perturbative side and bring our results into agreement with perturbation theory. Our findings will surely have consequences for the subtraction scheme used in such considerations.

There exist of course other methods which allow to extract the ultraviolet central charge from a massive integrable model. The c-theorem [36] has turned out to be very efficient in this context and a direct comparison is very suggestive. As the main input, the c-theorem requires the n-particle form factors (the 2-particle form factor is usually sufficient) related to the trace of the energy-momentum tensor. Unlike in the situation of conformal invariance, in which the trace vanishes, this tensor is not unique and acquires some scaling behaviour for massive theories. As observed in [38], the consequence of this fact is that the c-theorem produces a whole ray of central charges greater than the rank of the underlying Lie algebra **g**. The natural question which arises is to identify the origin of this ambiguity inside the TBA-approach. One might assume that it results from several possible solutions of the TBA-equation. However, this possibility is definitely ruled out as follows from our investigations in section 5. Since the statistical interaction $g$ enters the analysis as a further parameter, this could provide a further possible mechanism which produces the observed values. As our investigations demonstrate, however, also this possibility can be excluded for certain, since different choices for $g$ only produce central charges smaller than the rank of **g**. In the light of this results we conjecture that the responsible mechanism is to include a non-vanishing chemical potential in the way as was already indicated by Yang and Yang [1]. To settle this question precisely requires more detailed investigations [37].

There are several further questions which should be addressed in order to complete the picture. It will certainly be interesting to obtain also the higher terms beyond the first leading order in (1) and to exploit further the Y-systems of section 4.4. as an alternative analytical approach. There exist interesting links between these systems and spectral functions [39], such that one can expect more exact and universal results to follow.

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