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Ionization Probabilities through ultra-intense Fields in the extreme Limit

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Abstract

We continue our investigation concerning the question of whether atomic bound states begin to stabilize in the ultra-intense field limit. The pulses considered are essentially arbitrary, but we distinguish between three situations. First the total classical momentum transfer is non-vanishing, second not both the total classical momentum transfer and the total classical displacement are vanishing together with the requirement that the potential has a finite number of bound states and third both the total classical momentum transfer and the total classical displacement are vanishing. For the first two cases we rigorously prove, that the ionization probability tends to one when the amplitude of the pulse tends to infinity and the pulse shape remains fixed. In the third case the limit is strictly smaller than one. This case is also related to the high frequency limit considered by Gavrila et al.

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Ionization probabilities of atomic systems in the presence of intense laser fields are in general poorly predicted. Intense means here that the field intensities are of comparable size in magnitude with the ionization energy of the potential and hence conventional perturbation theory ceases to be valid. Numerous different methods for theoretical investigations have been carried out in order to treat the new intensity regime, such as perturbative methods around the Gordon-Volkov solution [1] of the Schrödinger equation [2, 3, 4, 5, 6, 7, 8], fully numerical solutions of the Schrödinger equation [9, 10, 11, 12, 13, 14, 15, 16], Floquet solutions [17, 18, 19], high frequency approximations [20] or analogies to classical dynamical systems [21]. Some of these investigations have led to the prediction of so-called atomic stabilization, which means that the ionization probability is supposed to decrease once a certain critical intensity has been surpassed. However, several authors have raised doubts and question whether such an effect really exists [7, 27, 25, 5, 8, 28, 29, 30]. For reviews on the subject we refer the reader to [22, 23, 24, 26].

In this paper we want to continue our previous investigations [28, 29, 30] and answer the question concerning the ionization probability in the limit when the field amplitude tends to infinity, while the pulse shape remains fixed. Of course strictly speaking one would have to include relativistic effects into the analysis at some high intensities and then a proper quantum field theoretical treatment is needed. However, the Schrödinger theory with the a.c. Stark Hamiltonian is consistent in itself, also in that regime, and in this light the limit becomes meaningful. Clearly our analysis does not capture the effect of window stabilization, which is the purported phenomenon that stabilization only occurs in a certain regime of high intensities and then the ionization probability tends to one once this regime is surpassed.

We consider the Schrödinger equation involving some potential $V(\vec{x})$, for instance the atomic potential, coupled to a classical linearly polarized electric field in the dipole approximation $E(t)$

$$i\frac{\partial\psi(\vec{x},t)}{\partial t} = \left(-\frac{\Delta}{2} + V(\vec{x}) + z\cdot E(t)\right)\psi(\vec{x},t) = H(t)\,\psi(\vec{x},t)\,. \qquad (1)$$

We use atomic units $\hbar = e = m_e = c\alpha = 1$ and we will mainly adopt the notations in [28]. We now want to state precisely which type of potentials and electric fields are included in our analysis.

Assumptions on V: *$V(\vec{x})$ is a real measurable function on $\mathbb{R}^3$. To each*

$\varepsilon > 0$ one may decompose V as

$$V = V_1 + V_2 \tag{2}$$

where V_1 is in $L^2(\mathbb{R}^3)$ (i.e. square integrable) with compact support and V_2 is in $L^\infty(\mathbb{R}^3)$ with

$$\|V_2\|_\infty = \operatorname{ess\,sup}_{\vec{x}\in\mathbb{R}^3} |V_2(\vec{x})| \le \varepsilon \quad . \tag{3}$$

Furthermore we assume that $H = H_0 + V$ with *$H_0 = -\frac{\Delta}{2}$ has no positive bound states.*

Relation (3) means that up to a set of measure zero $V_2(\vec{x})$ is bounded in absolute value by ε. We note that the potentials of atoms or molecules arising from Coulomb pair interactions belong to this wide class. To obtain for instance the decomposition (2) for the Coulomb potential $1/|\vec{x}|$ we set

$$\frac{1}{|\vec{x}|} = \frac{\chi_{1/\varepsilon}(\vec{x})}{|\vec{x}|} + \frac{1-\chi_{1/\varepsilon}(\vec{x})}{|\vec{x}|},$$

where $\chi_R(\vec{x})$ is the characteristic function of the ball $\{\vec{x}: \ |\vec{x}| \le R\}$ of radius R,

$$\chi_R(\vec{x}) = \begin{cases} 1 & \text{for} \quad |\vec{x}| \le R \\ 0 & \text{for} \quad |\vec{x}| > R \ . \end{cases}$$

Potentials satisfying the above assumptions are Kato small, i.e. for each α with $0 < \alpha \le 1$, there exists a constant $\beta = \beta(\alpha) \ge 0$ such that

$$\|V\psi\| \le \alpha \left\| -\frac{\Delta}{2}\psi \right\| + \beta \|\psi\| \tag{4}$$

for all $\psi \in L^2(\mathbb{R}^3)$ with $\Delta\psi \in L^2(\mathbb{R}^3)$. The Hamiltonian H is self-adjoint on the Hilbertspace $L^2(\mathbb{R}^3)$ and the domains $\mathcal{D}(H)$ and $\mathcal{D}(H_0)$ of definition of H and H_0 agree [31]. H is bounded from below and has no positive eigenvalues if V decays suitably at infinity [32, 33].

As for the conditions on the electric field, we assume that it takes on the form

$$E(t) = E_0 f(t), \tag{5}$$

where $f(t)$ is assumed to be measurable in t with $f(t)=0$ unless $0 \leq t \leq \tau$. We call τ the pulse duration, $f(t)$ the pulse shape and E_0 the amplitude of the pulse $E(t)$. Further we introduce the quantities

$$\begin{aligned} b(t) &= \int_0^t ds\, E(s) = E_0 \int_0^t ds\, f(s) = E_0 b_0(t) && (6) \\ c(t) &= \int_0^t ds\, b(s) = t b(t) - \int_0^t ds\, E(s)\, s = E_0 c_0(t) \quad . && (7) \end{aligned}$$

With e_z being the unit vector in z-direction, $b(\tau)\ e_z$ is the total classical momentum transfer and $c(\tau)\, e_z$ the total classical displacement. We are now in the position to formulate more precisely our assumptions on the electric field, that is on the pulse shape $f(t)$.

Assumptions on E: *$f(t)$ is a real measurable non-vanishing function in t, with support in the interval $[0,\tau]$ such that*

$$b_0(\tau) \neq 0. \tag{8}$$

In case the potential possesses a finite number of bound states we only assume that

$$b_0(\tau)^2 + c_0(\tau)^2 \neq 0. \tag{9}$$

Finally, $c_0(t)$ is supposed to be piecewise continuous possibly with a finite number of zeros in $[0,\tau]$.

Of course the restrictions of a finite number of bound states excludes the Coulomb Potential. However, we would like to remark that in general most numerical calculations in this context implicitly also assume a finite number of bound states. When projecting on bound states numerically, one is always forced to introduce a cut-off. Hence our analysis allows also in that case a direct comparison with such computations. The gain in the latter case is that when the requirement (8) is relaxed to (9) it allows to include more types of pulses such as Gaussian etc. All pulses used in the literature satisfy assumption E.

The ionization probability for any given normalized bound state ψ of the Hamiltonian H is given by

$$\mathcal{P}(\psi) = \|(1-P)\, U(\tau,0)\, \psi\|^2 = 1 - \|P U(\tau,0)\, \psi\|^2 \quad . \tag{10}$$

Here $U(t', t)$ denotes the unitary time evolution operator from time t to time t' associated to $H(t)$. Its existence [1] follows from results in [38, 39] (for details see [37]).

Further P denotes the orthogonal projection in $L^2(\mathbb{R}^3)$ on the space spanned by the bound states of $H = H_0 + V$. For more details on the precise definition of the ionization probability and its properties we refer the reader to [28]. In what follows $f(t)$ and V will be fixed.

We now formulate the Main Theorem of this article:

Theorem 1 *With the above assumptions on the electric field E and the potential V, the ionization probability $\mathcal{P}$ for any bound state ψ of $H = H_0 + V$ tends to one for the field amplitude E_0 going to infinity*

$$\lim_{|E_0|\to\infty} \mathcal{P}(\psi) = 1 \quad .$$

This improves a previous result in [28] (see relation (3.31) therein), which stated that $\lim_{|E_0|\to\infty} \mathcal{P}(\psi) \geq 1 - \tau^2 c$, where c is a constant depending on the potential V and on ψ only. The proof of this main theorem in case of condition (9) shows that the finite dimensional projector P may in fact be chosen arbitrarily. In particular in the case of the Coulomb potential, P may be the projector on the space spanned by *any* finite set of bound states.

Proof of the Main Theorem:

To start the proof of the main theorem following [28] we may first rewrite the ionization probability as

$$\mathcal{P}(\psi) = 1 - \left\| P \exp -ib(\tau)\, z \cdot \exp ic(\tau)\, p_z \cdot U'(\tau, 0)\, \psi \right\|^2 . \tag{11}$$

Here $U'(t', t)$ is the unitary time evolution operator associated to the Stark Hamiltonian (1) in the Kramers-Henneberger gauge [34, 35]

$$H'(t) = -\frac{\Delta}{2} + V(\vec{x} - c(t)\, e_z) \quad . \tag{12}$$

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[1] In order to show the existence one actually has to make some additional sufficient assumptions on V, namely one assumes that $V_i(\vec{x} - ue_z)$, $i = 1, 2$ are L^2 and L^∞ valued continuous functions in u, such that in addition

$$W_i(u) = \frac{\partial}{\partial u} V_i(\vec{x} - ue_z), \ i = 1, 2$$

exists and satisfies $\|W_1(u)\|_p < \infty$ for some $6/5 < p \leq 4/3$ and $\|W_2(u)\|_\infty < \infty$ uniformly in u on compact sets in $\mathbb{R}$. So strictly speaking we have to extend our assumptions on V. However, for standard potentials like Coulomb etc. this additional assumption is always satisfied and we therefore omitted it above for the sake of clarity.

Crucial for the proof of the main theorem will be the next result, that in the limit $E_0 \to \infty$ the time evolution U' for $H'(t)$ is just the free time evolution. We will need this result in the following form

Theorem 2 *For all* $\varphi \in L^2(\mathbb{R}^3)$

$$\lim_{|E_0|\to\infty} \|(U'(\tau,0) - \exp -i\tau H_0)\,\varphi\| = 0, \tag{13}$$

i.e. $U'(\tau,0)$ *converges strongly to* $\exp(-i\tau H_0)$ *as* $|E_0| \to \infty$.

The proof of this theorem will proceed in several steps. Before we begin with the proof we note that this is essentially Kato's theorem on the strong convergence of propagators for time dependent Hamiltonians [40]. However we cannot use this theorem directly since it is not valid for Hamiltonians with Coulomb interaction. Since $\|(U'(\tau,0) - \exp -i\tau H_0)\,\varphi\| \leq 2\,\|\varphi\|$ it suffices to prove (13) for all $\varphi \in \mathcal{D}(H_0) = \mathcal{D}(H)$, which is a dense set in $L^2(\mathbb{R}^3)$.

First we use Du Hamel's formula to write

$$(U'(\tau,0) - \exp -i\tau H_0)\,\varphi = -i\int_0^\tau U'(\tau,s)V(\vec{x} - c(s)e_z)\exp -isH_0\ \varphi ds \tag{14}$$

with $\varphi \in \mathcal{D}(H_0)$. We note that by the spectral theorem $\exp -isH_0$ leaves $\mathcal{D}(H_0)$ invariant. Therefore from (14) it follows that

$$\|(U'(\tau,0) - \exp -i\tau H_0)\,\varphi\| \leq \int_0^\tau \|V(\vec{x} - c(s)e_z)\exp -isH_0 \cdot \varphi\|\,ds, \tag{15}$$

uniformly in E_0.

To proceed further we use the following

Lemma 3 *For any* $\varphi \in \mathcal{D}(H) = \mathcal{D}(H_0)$ *and all* $s \in [0,\tau]$ *with* $c_0(s) \neq 0$ *one has*

$$\lim_{E_0\to\infty} \|V(\vec{x} - c(s)\gamma e_z)\,\varphi\| = 0. \tag{16}$$

Proof: It suffices to show that for any $\varphi \in \mathcal{D}(H) = \mathcal{D}(H_0)$

$$\lim_{|\gamma| \to \infty} \|V(\vec{x} - \gamma e_z)\varphi\| = 0.$$

We show that for arbitrary small $\varepsilon > 0$ the estimate $\|V(\vec{x} - \gamma e_z)\varphi\| < \varepsilon$ holds for all sufficiently large $\gamma > 0$.

Since V is Kato small and since $-\Delta$ commutes with translations, the potential in the Kramers-Henneberger gauge satisfies a similar estimate

$$\|V(\vec{x} - \gamma e_z)\varphi\| \leq \left\| -\frac{\Delta}{2}\varphi \right\| + \beta \|\varphi\| \tag{17}$$

with fixed $\beta < \infty$ and for all γ.

Indeed,

$$\begin{aligned} \|V(\vec{x} - \gamma e_z)\varphi\| &= \|V(\vec{x}) \exp(i\gamma p_z)\ \varphi\| \\ \leq \|H_0 \exp(i\gamma p_z)\ \varphi\| + \beta \|\exp(i\gamma p_z)\ \varphi\| &= \|H_0\varphi\| + \beta\|\varphi\|, \end{aligned}$$

where the last equality follows from the fact that H_0 commutes with the translations. In comparison with (4) we have taken $\alpha = 1$ and chosen $\beta = \max(\beta(\alpha = 1), 1)$. Hence it suffices to prove (16) on a core for H_0 which is also a core for H. We recall that $\mathcal{C}$ is a core for a self-adjoint operator A with domain $\mathcal{D}(A)$, if $\mathcal{C}$ is contained and dense in $\mathcal{D}(A)$ with respect to the topology in $\mathcal{D}(A)$ given by the norm $\|\varphi\|_{\mathcal{D}(A)} = \|A\varphi\| + \|\varphi\|$. Indeed, for a given $\varphi \in \mathcal{D}(H_0)$ let $\varphi' \in \mathcal{C}$ be such that

$$\|H_0(\varphi - \varphi')\| + \|(\varphi - \varphi')\| \leq \frac{\varepsilon}{2\beta}\ .$$

Also let $\gamma(\varepsilon, \varphi')$ be such that

$$\|V(\vec{x} - \gamma e_z)\varphi'\| \leq \frac{\varepsilon}{2} \qquad \text{for all} \qquad \gamma \geq \gamma(\varepsilon, \varphi').$$

Then

$$\begin{aligned} \|V(\vec{x} - \gamma e_z)\varphi\| &\leq \|V(\vec{x} - \gamma e_z)(\varphi - \varphi')\| + \|V(\vec{x} - \gamma e_z)\varphi'\| \\ &\leq \|H_0(\varphi - \varphi')\| + \beta \|(\varphi - \varphi')\| + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Now $C_0^\infty(\mathbb{R}^3)$, the set of smooth functions on $\mathbb{R}^3$ with compact support, is such a core and we will now prove (16) on this core. Assuming that $\varphi \in C_0^\infty(\mathbb{R}^3)$ is normalized, we obtain with the assumptions on V

$$\begin{aligned} \|V(\vec{x}-\gamma e_z)\varphi\| &\leq \|V_1(\vec{x}-\gamma e_z)\varphi\| + \|V_2(\vec{x}-\gamma e_z)\varphi\| \\ &\leq \|V_1(\vec{x}-\gamma e_z)\varphi\| + \varepsilon. \end{aligned}$$

For $|\gamma|$ sufficiently large $V_1(\vec{x}-\gamma e_z)\varphi = 0$ and the lemma follows.

We proceed with the proof of the theorem. Since for fixed $\varphi \in \mathcal{D}(H_0)$ the map $\exp(-isH_0) : [0,\tau] \to \mathcal{D}(H_0)$ given by $s \mapsto \exp(-isH_0)\varphi$ is continuous, the set $\mathcal{S} = \{\exp(-isH_0)\varphi\}_{0\leq s\leq\tau}$ is compact in $\mathcal{D}(H_0)$. Therefore

$$\|V(\vec{x}-c(s)e_z)\psi\| \to 0$$

as $|E_0| \to \infty$ uniformly in $\psi \in \mathcal{S}$ for all $s \in [0,\tau]$ except the finite set where $c_0(s) = 0$ (see e.g. [41]). Now the r.h.s. of (15) for any $\varphi \in \mathcal{D}(H_0)$ can be bounded by

$$\|(U'(\tau,0) - \exp -i\tau H_0)\varphi\| \leq \int_0^\tau \sup_{\psi\in\mathcal{S}} \|V(\vec{x}-c(s)e_z)\psi\| ds. \tag{18}$$

From (17) and the definition of $\mathcal{S}$ it follows also that

$$\|V(\vec{x}-c(s)e_z)\psi\| \leq C_\varphi$$

for all $\psi \in \mathcal{S}$ and all $s \in [0,\tau]$ with

$$C_\varphi = \left\| -\frac{\Delta}{2}\varphi \right\| + \beta \|\varphi\|$$

for every $\varphi \in \mathcal{D}(H_0)$ uniformly in E_0.

By the Lebesgue dominated convergence theorem we therefore have that the right hand side of (18) tends to zero as $|E_0| \to \infty$.

This completes the proof of the theorem.

Remark 5 From the preceding discussion, it is obvious how to weaken the last condition in the assumptions on E. Assume we may divide the interval $[0,\tau]$ into $2N+1$ parts as $0 = \tau_0 < \tau_1 < \cdots < \tau_{2N} < \tau_{2N+1} = \tau$,

> such that $c_0(t)$ vanishes identically in the intervals $[\tau_{2j}, \tau_{2j+1}]$ $(0 \leq j \leq N)$ and is non-zero except for a finite set in the intervals $[\tau_{2j+1}, \tau_{2j+2}]$ $(0 \leq j \leq N-1)$. Then $U'(\tau, 0)$ converges strongly to
>
> $$U'' = e^{-i(\tau_{2N+1}-\tau_{2N})H_0} \cdot e^{-i(\tau_{2N}-\tau_{2N-1})H} \cdots e^{-i(\tau_2-\tau_1)H} \cdot e^{-i(\tau_1-\tau_0)H_0} \quad . \tag{19}$$
>
> Since $f(t)$ is by assumption not identically zero, $c_0(t)$ is not identically zero on $[0, \tau]$, so $U'' \neq \exp -i\tau H$.

To prove the main theorem it suffices now by Theorem 2 and the obvious estimate

$$\begin{aligned} & \left\| Pe^{-ib(\tau)z} \cdot e^{ic(\tau)p_z} \cdot U'(\tau,0)\psi \right\| \\ \leq \; & \left\| Pe^{-ib(\tau)z} \cdot e^{ic(\tau)p_z} \cdot e^{-i\tau H_0}\psi \right\| + \left\| Pe^{-ib(\tau)z} \cdot e^{ic(\tau)p_z} \cdot \left(U'(\tau,0) - e^{-i\tau H_0}\right)\psi \right\| \\ \leq \; & \left\| Pe^{-ib(\tau)z} \cdot e^{ic(\tau)p_z} \cdot e^{-i\tau H_0}\psi \right\| + \left\| \left(U'(\tau,0) - e^{-i\tau H_0}\right)\psi \right\| \end{aligned}$$

to show that

$$\lim_{|E_0|\to\infty} \left\| P \exp -ib(\tau) z \cdot \exp ic(\tau) p_z \cdot \exp -i\tau H_0 \psi \right\| = 0 \quad . \tag{20}$$

Here $\exp -i\tau H_0$ has to be replaced by U'' in case remark 5 applies. Since $\exp -i\tau H_0$ leaves $\mathcal{D}(H_0)$ invariant, it is enough to show

$$\lim_{|E_0|\to\infty} \left\| P \exp -ib(\tau) z \cdot \exp ic(\tau) p_z \varphi \right\| = 0 \quad . \tag{21}$$

for all $\varphi \in \mathcal{D}(H) = \mathcal{D}(H_0)$ in order to prove (20).

We now modify some arguments already used in [36] and [28]. First we consider the case when $b_0(\tau) \neq 0$. Also by assumption we have $PH \leq 0$. Hence $P\left(H - \frac{1}{2}b(\tau)^2\right)^{-1}$ is a well defined operator with norm smaller or equal to $2/b(\tau)^2$. Therefore we have

$$\begin{aligned} & \left\| P \exp -ib(\tau) z \cdot \exp ic(\tau) p_z \varphi \right\| \\ & = \left\| P\left(H - \frac{1}{2}b(\tau)^2\right)^{-1} \left(H - \frac{1}{2}b(\tau)^2\right) \exp -ib(\tau) z \cdot \exp ic(\tau) p_z \varphi \right\| \\ & \leq \frac{2}{b(\tau)^2} \left\| \left(H - \frac{1}{2}b(\tau)^2\right) \exp -ib(\tau) z \cdot \exp ic(\tau) p_z \varphi \right\| \quad . \end{aligned} \tag{22}$$

Inserting the relation

$$\begin{aligned} & \exp -ic\left(\tau\right)p_z\cdot\exp ib\left(\tau\right)z\cdot H\cdot\exp -ib\left(\tau\right)z\cdot\exp ic\left(\tau\right)p_z \\ = \quad & H_0-b\left(\tau\right)p_z+\frac{1}{2}b\left(\tau\right)^2+V\left(\vec{x}-c\left(\tau\right)e_z\right) \end{aligned} \tag{23}$$

into (22) yields

$$\begin{aligned} & \left\|P\exp -ib\left(\tau\right)z\cdot\exp ic\left(\tau\right)p_z\varphi\right\| \\ \leq \quad & \frac{2}{b\left(\tau\right)^2}\left\{\left\|H_0\varphi\right\|+b\left(\tau\right)\left\|p_z\varphi\right\|+\left\|V\left(\vec{x}-c\left(\tau\right)e_z\right)\varphi\right\|\right\}\ . \end{aligned} \tag{24}$$

Furthermore

$$\left\|p_z\varphi\right\|^2=\left\langle\varphi,p_z^2\varphi\right\rangle\leq 2\left\langle\varphi,H_0\varphi\right\rangle\leq\left\langle\varphi,{H_0}^2\varphi\right\rangle+\left\langle\varphi,\varphi\right\rangle,$$

such that

$$\left\|p_z\varphi\right\|\leq\left\|H_0\varphi\right\|+\left\|\varphi\right\|\ . \tag{25}$$

Finally we have that $\left\|V\left(\vec{x}-c\left(\tau\right)e_z\right)\varphi\right\|$ is uniformly bounded in E_0 by Lemma 2 and therefore we may control the limit $|E_0|\to\infty$ in (24), i.e. the r.h.s. goes as $\mathcal{O}\left(\frac{1}{\|E_0\|}\right)$. This concludes the proof of the Main Theorem for the case $b_0\left(\tau\right)\neq 0$.

We now turn to the case when P is a finite dimensional projection and $b_0\left(\tau\right)^2+c_0\left(\tau\right)^2 =: a_0^2\neq 0$. Actually by what has already been proved, it would suffice to consider the case $b_0(\tau)=0, c_0(\tau)\neq 0$ only. However, we will prove the claim for an arbitrary finite dimensional P not necessarily being the projection onto the space spanned by the bound states of H.

We start with two preliminary considerations. First, by the Campbell-Hausdorff formula

$$\exp -ib\left(\tau\right)z\cdot\exp ic\left(\tau\right)p_z=\exp\frac{i}{2}b\left(\tau\right)c\left(\tau\right)\cdot\exp i\left(c\left(\tau\right)p_z-b\left(\tau\right)z\right)\ . \tag{26}$$

Now there is always an s such that

$$c\left(\tau\right)p_z-b\left(\tau\right)z=E_0a_0\left(z\cos s+p_z\sin s\right)=E_0a_0Z(s)\ . \tag{27}$$

Introducing the unitary operator $W(s)=\exp\frac{is}{2}\left(p_z^2+z^2\right)$ we may perform a Bogoliubov transformation on z

$$W(s)\,z\,W(s)^{-1}=Z(s)\ . \tag{28}$$

Secondly, let $\varphi_n (1 \leq n \leq N)$ be an orthonormal basis for the range of P. Then

$$\begin{aligned} & \left\| P \exp -ib\left(\tau\right) z \cdot \exp ic\left(\tau\right) p_z \varphi \right\|^2 \\ = & \sum_{n=1}^{N} \left| \langle \varphi_n, \exp iE_0 a_0 Z(s) \cdot \varphi \rangle \right|^2 \\ = & \sum_{n=1}^{N} \left| \left\langle W(s)^{-1} \varphi_n, \exp iE_0 a_0 z \cdot W(s)^{-1} \varphi \right\rangle \right|^2 \\ = & \sum_{n=1}^{N} \left| \int d\vec{x} \overline{\left(W(s)^{-1}\varphi_n\right)(\vec{x})} \left(W(s)^{-1}\varphi\right)(\vec{x}) \exp iE_0 a_0 z \right|^2 \quad . \end{aligned}$$

Since $\overline{\left(W(s)^{-1}\varphi_n\right)(\vec{x})} \left(W(s)^{-1}\varphi\right)(\vec{x}) \in L^1\left(\mathbb{R}^3\right)$ the right hand side of the last equation vanishes in the limit $|E_0| \to \infty$ by the Riemann-Lebesgue theorem, which concludes the proof of the Main Theorem.■

Now we turn to the case when $b_0\left(\tau\right) = c_0\left(\tau\right) = 0$. Notice, that if we consider linearly polarized light, then for the most common pulse shapes like for instance a static envelope, trapezoidal envelope, sine-squared envelope etc. [29] the extreme high frequency limit, i.e. $\omega \to \infty$ leads to $b_0\left(\tau\right) = c_0\left(\tau\right) = 0$. This limit is needed in order to apply the analysis of Gavrila and coworkers [20], which provides so far the most profound "explanation" for the occurrence of stabilization.

We prove the Second Main Theorem:

Theorem 4 *Let $b_0\left(\tau\right) = c_0\left(\tau\right) = 0$. Denote*

$$p(\tau) = \lim_{|E_0| \to \infty} \mathcal{P}\left(\psi\right) = \left\| (1-P) \exp\ -i\tau H_0 \cdot \psi \right\|^2 . \qquad (29)$$

If H has only one bound state (i.e. if P is one dimensional), then $p(\tau) > 0$ for all $\tau > 0$.

Furthermore $p(\tau) < 1$ (at least) for all $\tau \in [0, \tau_*]$, where

$$\tau_* = \pi \left[\langle H_0 \psi, H_0 \psi \rangle - \langle \psi, H_0 \psi \rangle^2 \right]^{-1/2} .$$

Proof: We consider the survival probability $q(\tau) = \left|(\psi, e^{-i\tau H_0}\psi)\right|^2$ and observe that $p(\tau) = 1 - q(\tau)$ if there is only one bound state. We prove that $q(\tau) < 1$ for all $\tau > 0$. Note that by Schwarz inequality the bound $q(\tau) \le 1$ is trivial. Now

$$\left(\psi, e^{-i\tau H_0}\psi\right) = \int_0^\infty e^{-i\lambda\tau} d\mu_\psi(\lambda) \equiv \widehat{\mu_\psi}(\tau),$$

where μ_ψ is the (nonnegative, absolute continuous) spectral measure associated with H_0,

$$\mu_\psi((-\infty, \lambda]) = \begin{cases} \int_{|\vec{p}| \le \sqrt{2\lambda}} |\widehat{\psi}(p)|^2 d\vec{p}, & \lambda \ge 0 \\ 0, & \lambda < 0. \end{cases}$$

Obviously, $\int_{\mathbb{R}} d\mu_\psi(\lambda) = 1$. It is well known (see e.g. [42]) that $|\widehat{\mu_\psi}(\tau)| < 1$ for all $\tau > 0$ when the measure μ_ψ is absolutely continuous.

The second part of the theorem follows from the estimate of Pfeifer [43]. ■

We note that due to the Paley-Wiener theorem $\widehat{\mu_\psi}(\tau)$ cannot have compact support. Therefore the inequality $p(\tau) < 1$ must be valid for some suitable arbitrary large $\tau > 0$. On the other hand, it is well known that $|\widehat{\mu_\psi}(\tau)| \le C\tau^{-N}$ for arbitrary N and all large $\tau > 0$, since the spectrum of H_0 is purely transient absolute continuous (see for instance [44]). This means that in case H has only one bound state, say, the ionization probability $p(\tau)$ will tend to one faster than any power inverse power of τ for $\tau \to \infty$.

Example 1 The easiest pulse shape for which theorem 4 applies is $f(t) = \cos(\omega t)$, since then $c_0(\tau = \frac{2\pi n}{\omega}) = b_0(\tau = \frac{2\pi n}{\omega}) = 0$. As a concrete example for the potential we choose the Coulomb potential $V(\vec{x}) = -1/|\vec{x}|$. The normalized wave function of the ground state in the momentum representation is given by (see for instance [45])

$$\Psi(\vec{p}) = \frac{\sqrt{8}}{\pi} \frac{1}{(1+p^2)^2} \quad ,$$

such that the survival probability in this case reads

$$q(\tau) = \left(\frac{32}{\pi}\right)^2 \left| \int\limits_0^\infty dp\, p^2 \frac{e^{-i\tau\frac{p^2}{2}}}{(1+p^2)^4} \right|^2 = \frac{64}{\pi} \left| U(\frac{3}{2}, -\frac{3}{2}; \frac{i\tau}{2}) \right|^2 \quad .$$

Here $U(a,b;z)$ denotes a confluent hypergeometric function (see for instance [46]). So for typical sub-picosecond pulses we obtain for instance $q(\tau = 400 a.u.) = 2.45\ 10^{-6}$ and $q(\tau = 1000 a.u.) = 1.63\ 10^{-7}$. Essential is here to note that the survival probability is always non-vanishing and monotonically decreasing in τ (see figure 1).

Example 2 We now take the potential to be the attracting point interaction, often also called the delta potential in three dimensions, (see e.g. [47]) with coupling constant $\alpha > 0$. This potential has the virtue that it possess only one eigenstate

$$\Psi(\vec{x}) = \sqrt{\frac{\alpha}{2\pi}} \frac{e^{-\alpha|\vec{x}|}}{|\vec{x}|}$$

with energy $-(\alpha)^2$. In the momentum representation the wave function is given by

$$\Psi(\vec{p}) = \frac{\sqrt{\alpha}}{\pi} \frac{1}{(\alpha^2 + p^2)}$$

such that the survival probability turns out to be

$$q_\alpha(\tau) = \frac{16\alpha^2}{\pi^2} \left| \int_0^\infty dp\, p^2 \frac{e^{-i\tau\frac{p^2}{2}}}{(\alpha^2+p^2)^2} \right|^2 = \frac{1}{\pi} \left| U(\frac{3}{2}, \frac{1}{2}; \frac{i\tau\alpha^2}{2}) \right|^2 \quad .$$

As figure 2 illustrates, the survival probability decreases monotonically with increasing α for fixed pulse duration τ. The figure also shows, that for increasing τ the survival probability decreases.

We may assume that the Hydrogen atom behaves with respect to the energy variation qualitatively the same way as the point interaction. Then this example indicates that one should expect that for sufficiently high Rydberg states the survival probability $q(\tau)$ will be sufficiently close to 1 even for times $\tau \approx 1$ ps.

Conclusion We have investigated the ionization probability in the extreme intensity limit for three different situations. The first analysis presumes that the classical momentum transfer $b_0(\tau)$ is non-vanishing and allows essentially all common potentials. Since the condition $b_0(\tau) \neq 0$ excludes a wide range of possible pulses, we also studied separately a situation for which we only demand that not both the classical momentum transfer $b_0(\tau)$ and the total classical momentum transfer $c_0(\tau)$ vanish simultaneously. In addition we have to demand for this case that potential only possess a finite number of bound states. This is similar to the situation in many numerical calculations, in which one is also forced to introduce a cut-off at some level when projecting onto bound states. In both cases we find that the ionization probability $\mathcal{P}$ for any bound state of the Hamiltonian $H = H_0 + V$ for the field amplitude E_0 going to infinity is going to one. This excludes in our opinion the possibility of stabilization for these situations, apart from window stabilization.

Finally, we considered the situation in which $b_0(\tau) = c_0(\tau) = 0$ and find indeed the possibility of stabilization. For the most common pulses, which involve linearly polarized light, this case corresponds to the high frequency limit of Gavrila and coworkers [20]. We conclude that our analysis is consistent with the "high frequency picture" and that stabilization is only to be expected in this latter case.

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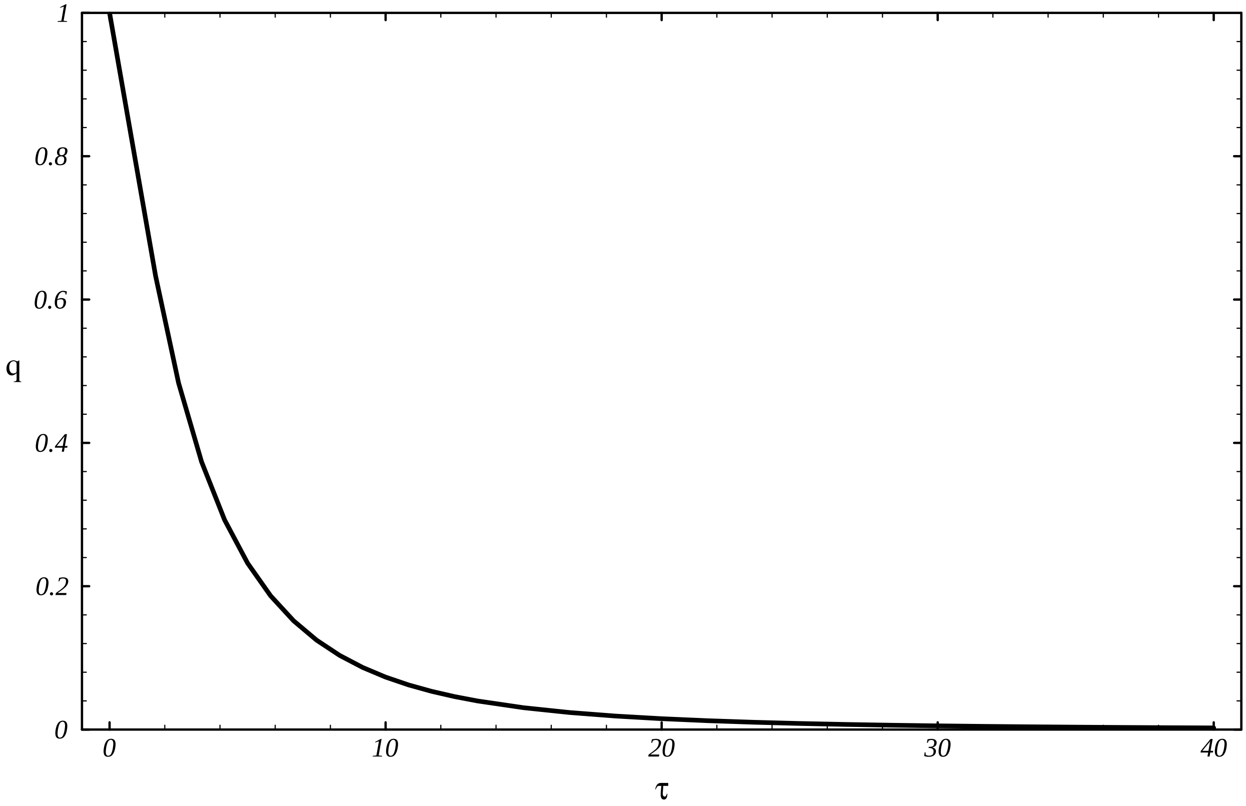
Figure 1: Survival probability $q(\tau(a.u.))$ after the time τ for the ground state of the Hydrogen atom under the free time evolution.

Figure 2: Survival probability $q_\alpha(\tau(a.u.))$ for fixed pulse duration $\tau = 200a.u.$ dotted line, $\tau = 400a.u.$ dashed line, $\tau = 1000a.u.$ solid line, for the bound state of the three dimensional delta potential under the free time evolution as a function of the coupling α.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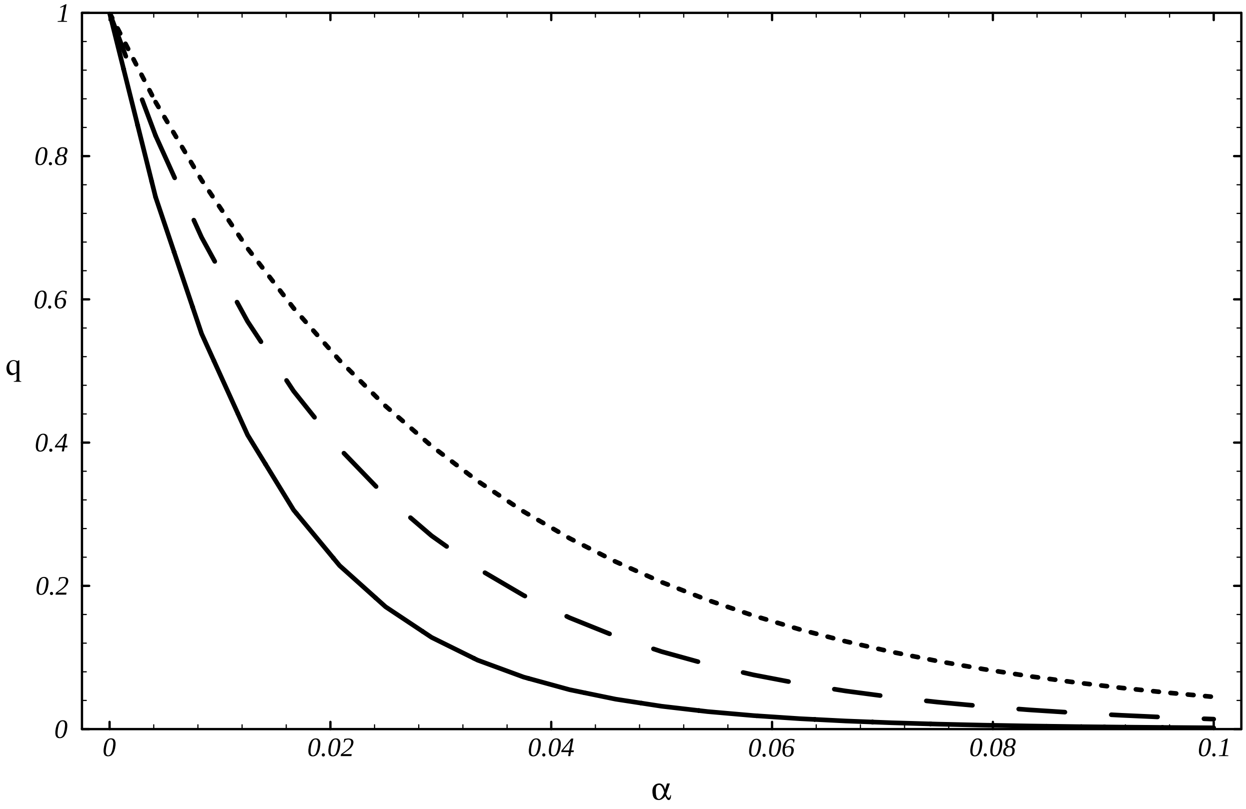
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