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Vertex Operators and Soliton Time Delays in Affine Toda Field Theory

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Abstract

In a space-time of two dimensions the overall effect of the collision of two solitons is a time delay (or advance) of their final trajectories relative to their initial trajectories. For the solitons of affine Toda field theories, the space-time displacement of the trajectories is proportional to the logarithm of a number X depending only on the species of the colliding solitons and their rapidity difference. X is the factor arising in the normal ordering of the product of the two vertex operators associated with the solitons. X is shown to take real values between 0 and 1. This means that, whenever the solitons are distinguishable, so that transmission rather than reflection is the only possible interpretation of the classical scattering process, the time delay is negative and so an indication of attractive forces between the solitons.

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1 Introduction

Affine Toda field theories [1] are relativistically invariant field theories which are integrable in a space-time of two dimensions and possess a natural interpretation as special deformations of conformally invariant theories [2, 3, 4]. When the coupling is imaginary so that there are degenerate vacua, the equations support solutions describing any number of solitons interpolating the vacua. A number of authors have worked out examples on a case by case basis [5, 6, 7, 8]. On the other hand, a general formalism for these solutions has recently been found [9, 10] exploiting a basis of the underlying affine Kac-Moody algebra in which the principal Heisenberg subalgebra plays a significant rôle. This subalgebra is isomorphic to the algebra of conserved charges or "energies" and can be thought of as an infinite Poincaré algebra appropriate to an integrable theory. The simplest such theory, namely that associated with affine $su(2)$, is very familiar as sine-Gordon theory [11, 12].

In the formalism, the individual solitons are "created" by group elements obtained by exponentiating quantities $\hat{F}^1, \hat{F}^2, \ldots, \hat{F}^r$ which ad-diagonalise the "energies" generating the Heisenberg subalgebra. Each exponential series terminates with the highest non-vanishing power of $\hat{F}^i$ being expressible as a vertex operator obtained by exponentiating and normal ordering an element of the Heisenberg subalgebra [10, 13] when the affine Kac-Moody algebra is untwisted and simply laced (and also when it is twisted [14]). This result is sufficient to show that these solutions correctly interpolate degenerate vacua.

In this paper we show that these vertex operators determine yet more detail of the asymptotic behaviour of the soliton solutions. In these solutions the energy-momentum vector of a specific soliton is unchanged by collision but the trajectory may sustain a lateral displacement in space-time as discussed in section 2. Traditionally this is parametrised by the time delay in the centre of momentum frame. After a review of the vertex operator formalism in section 3, our first result, in section 4, is that this lateral displacement can be expressed straightforwardly in terms of the logarithms of the numbers $X_{ik}(\theta_i - \theta_k)$ arising in the procedure of normal ordering the product of two vertex operators mentioned above as being associated with solitons i and k.

In section 5 the overall lateral displacement of the soliton trajectories due to the scattering with several other solitons is determined and shown to be independent of the temporal order in which the collisions take place. This is because the displacement is simply additive. The result constitutes the classical analogue of the Yang-Baxter equation [15] and bootstrap equations [16] for the quantum scattering matrix featuring the factorisation property of the n particle S-matrix into two particle S-matrices.

In section 6 various properties of the number $X_{ik}(\theta_i - \theta_k)$ are established, including symmetry and crossing properties. In particular it is verified that it is real when the rapidity difference is real, as the physical interpretation demands. Furthermore it is shown to take values restricted to lie between 0 and 1, so that the associated time delay (in the centre of momentum frame) is always negative. Suppose two distinguishable solitons are considered in the sense that they carry different species or different topological charges. In this case the solution describing the scattering has to be regarded as a transmission rather than a reflection. If the solitons are indistinguishable either interpretation is possible. With this understanding our result indicates that the forces between two distinguishable solitons are always attractive because of the time advance.

In the concluding section 7, we mention the well known connection between the time delay and the semi-classical approximation to the S-matrix as well as the intriguing similarity of the structure of $X_{ik}(\theta_i - \theta_k)$ with the known scattering matrices in affine Toda field theories [17, 18].

2 Kinematics of scattering of two particles in two dimensions

In an integrable theory in two dimensions, when two particles collide the outcome consists of two particles with the same masses as the original particles. If the two masses differ, the corresponding energy-momentum vectors are unchanged. If the two masses are equal, even though the particles are distinct, it is kinematically possible for the energy-momentum vectors to interchange.

Classically the particles describe trajectories in space-time which are straight except near the collision. The collision may displace the trajectories laterally but as the energy-momentum is unchanged, the final direction coincides with the initial direction. We now consider alternative descriptions of the displacements of the trajectories and show how the conservation laws correlate the displacement of the trajectories of the two colliding particles.

Consider first a single particle with velocity v, energy E, and hence momentum vE. Before collision the equation of the trajectory in space-time is

$$x = vt + x(I), \tag{2.1}$$

whereas afterwards it is

$$x = vt + x(F), \tag{2.2}$$

as the velocity is unchanged. Only the intercept with the x-axis changes. So

$$\Delta(x) = x(F) - x(I) \tag{2.3}$$

measures the lateral displacement at fixed time. This is not Lorentz invariant but the combination $E\Delta(x)$ is. Since E is always positive it follows that $\Delta(x)$ has the same sign in all Lorentz frames of reference, even though its magnitude varies. The intercepts of the trajectories (2.1) and (2.2) with the time axis are given by

$$t(I) = -\frac{x(I)}{v}, \qquad t(F) = -\frac{x(F)}{v}. \tag{2.4}$$

Then, we define the "time delay"

$$\Delta(t) = t(F) - t(I) = -\frac{\Delta(x)}{v}. \tag{2.5}$$

Again

$$E\Delta(x) = -p\Delta(t) \tag{2.6}$$

is Lorentz invariant. As the sign of p can be changed by a Lorentz transformation so can that of $\Delta(t)$.

Now consider both particles participating in the collision, labelling them 1 and 2. Consider the "centre of energy" coordinate

$$X = \frac{E_1 x_1 + E_2 x_2}{E_1 + E_2}.$$

Then

$$\frac{dX}{dt} = \frac{p_1 + p_2}{E_1 + E_2}$$

is constant throughout time so that

$$X = \frac{p_1 + p_2}{E_1 + E_2} t + X_0.$$

Now compare the results of inserting the trajectories (2.1) before the collision with the result of inserting (2.2) after the collision. As the results must agree

$$E_1\Big(x_1(F) - x_1(I)\Big) + E_2\Big(x_2(F) - x_2(I)\Big) = 0$$

or, denoting $\Delta_{12}(x) = x_1(F) - x_1(I)$ and similarly for $\Delta_{21}(x)$

$$E_1\Delta_{12}(x) + E_2\Delta_{21}(x) = 0. \tag{2.7}$$

Thus the spatial displacement of the two trajectories must have opposite signs. By (2.6) we have, equally,

$$p_1\Delta_{12}(t) + p_2\Delta_{21}(t) = 0, \tag{2.8}$$

where $\Delta_{12}(t) = t_1(F) - t_1(I)$ is the time delay sustained by particle 1 colliding with particle 2. Notice that, in the centre of momentum frame $p_1 + p_2 = 0$, the two time delays are equal:

$$\Delta_{12}(t) = \Delta_{21}(t). \tag{2.9}$$

We see from (2.6), (2.7) and (2.8) that if particle 1 moves faster than particle 2, then the three quantities (2.9), $\Delta_{21}(x)$ and $-\Delta_{12}(x)$ all have the same sign. This common sign has a physical interpretation. Suppose the force between the particles is attractive. Then particle 1 will accelerate as it approaches particle 2 and afterwards decelerate. As a result $\Delta_{12}(x)$ will be positive and the common sign negative. Thus an attractive force implies a negative time delay, in other words a time advance, in the centre of momentum frame. A repulsive force implies a time delay, if there is transmission. There is also the additional possibility of a reflection with either a delay or advance if the two masses are equal.

The preceding discussion of relativistic particles colliding classically applies also to relativistic solitons and, in particular, to the solitons of affine Toda field theory. In the subsequent sections we shall show that when soliton 1 collides with soliton 2 the resultant displacements are given by

$$E_1\Delta_{12}(x) = -p_1\Delta_{12}(t) = -\mathrm{sign}(p_1 - p_2)\frac{2h}{|\beta|^2}\ln X_{12}(\theta_1 - \theta_2) \tag{2.10}$$

where h is the Coxeter number associated with the theory and $|\beta|$ the magnitude of the imaginary coupling constant β. The quantity $X_{12}(\theta_1 - \theta_2)$ depending on the rapidity difference $\theta = \theta_1 - \theta_2$ of the two solitons has been met before. It occurs when the product of the two vertex operators associated with the solitons 1 and 2 are normal ordered [10]. Equation (2.10) generalises the well known result for the time delay in sine-Gordon theory [19, 20] when

$$X_{12}(\theta) = \tanh^2\left(\frac{\theta}{2}\right).$$

Thus $X_{12}(\theta)$ has acquired a new physical interpretation whose viability requires it to possess various properties not hitherto apparent. For example we shall show that $X_{12}(\theta)$ is real when θ is real and that it satisfies the symmetry property

$$X_{12}(\theta) = X_{21}(\theta)$$

demanded by (2.7). Furthermore it takes values between 0 and 1. Hence, in the centre of momentum frame, the time delay is always negative by (2.10) whatever the velocities concerned. As explained, this suggests that affine Toda solitons exert attractive forces on each other. The only possible exception to this is when two identical solitons are considered, with the same species and topological quantum number. Then it is possible to interpret the scattering as a reflection rather than a transmission. In this case it is possible for the force to be repulsive. This is the accepted point of view in sine-Gordon theory which furnishes a special case of our result (2.10) [21].

3 Soliton solutions and vertex operators

Here we shall recall the general formalism for soliton solutions in affine Toda field theory and the rôle played by vertex operators, at least when the associated affine Kac-Moody algebra is untwisted and simply laced. The extension to the twisted case is straightforward in view of the work of [14] and to the untwisted non simply laced case only slightly more complicated.

When the coupling constant β is purely imaginary the affine Toda field theories possess classical solutions describing any number, N, say of solitons which may be composed of any of the r = rank g species (where $\hat{g}$ is the associated affine Kac-Moody algebra). The solution takes the form

$$e^{-\beta\lambda_j.\phi} = \frac{\langle\Lambda_j|g(t)|\Lambda_j\rangle}{\langle\Lambda_0|g(t)|\Lambda_0\rangle^{m_j}}. \tag{3.1}$$

$\phi(x,t)$ is the r component affine Toda field. λ_j is the j^{th} fundamental weight of the finite dimensional Lie algebra g, while Λ_j is the corresponding weight of $\hat{g}$, following the notation of [22]. $|\Lambda_j\rangle$ is the highest weight of the corresponding "highest weight" representation whose level is m_j. Λ_0 denotes the zero-th fundamental weight of $\hat{g}$. $|\Lambda_0\rangle$ can be regarded as the vacuum state at level $m_0 = 1$. The Kac-Moody group element $g(t)$ in (3.1) contains the soliton data: it factorises into N factors, each one characteristic of each individual soliton

$$g(t) = g_N(t)g_{N-1}(t)\cdots g_1(t) \tag{3.2}$$

where

$$g_m(t) = e^{Q_m W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)} \tag{3.3}$$

are the factors in (3.2) and ordered according to the rapidities θ_m. The real functions $W_{i(m)}(\theta_m)$ carry the dependence on the space, x, and time, t, variables:

$$W_{i(m)}(\theta_m) = e^{\mu_{i(m)}(x\cosh\theta_m - t\sinh\theta_m)}. \tag{3.4}$$

The m^{th} soliton has "species" $i(m)$ and rapidity θ_m. When quantised, the affine Toda field ϕ creates r species of particles whose masses are $\hbar\mu_1, \hbar\mu_2, \ldots, \hbar\mu_m$. Thus (3.4) provides a precise correspondence between the r species of solitons and the r species of field excitation particle. When g is simply laced so that all roots can be taken to have length $\sqrt{2}$, the ratios of the masses of corresponding soliton and field excitation particle are independent of the species i. It has been shown [5, 9] that the mass of the i'th species of soliton

$$M_i = \frac{2h\mu_i}{|\beta|^2}. \tag{3.5}$$

A similar result holds for the twisted theories[14]. The quantities $\hat{F}^i(\theta)$ are generators of $\hat{g}$ which ad-diagonalise the principal Heisenberg subalgebra

$$\left[\hat{E}_M, \hat{F}^i(\theta)\right] = \gamma_i \cdot q([M])(z_i)^M \hat{F}^i(\theta) \tag{3.6}$$

in the notation of [9]. The elements of the principal Heisenberg subalgebra are graded by $d' = T^3 - hL_0$, the "principal" grade:

$$\left[\hat{E}_M, \hat{E}_N\right] = xM\delta_{M+N,0} \qquad , \qquad \left[d', \hat{E}_M\right] = M\hat{E}_M. \tag{3.7}$$

Here x is the level of the representation considered and it is understood that the M can only equal an exponent of $\hat{g}$ that is an exponent of g modulo its Coxeter number h. The complex number $z_{i(m)}$ in (3.6) is related to the rapidity θ_m by

$$z_{i(m)} = ie^{-\theta_m} e^{-i\pi \frac{(1+c(i))}{2h}} \tag{3.8}$$

where the phase ensures that $W_{i(m)}$ is real. The complex number Q_m can be parametrised as

$$Q_m = e^{i\psi_m} e^{-\mu_{i(m)} x_m^0 \cosh\theta_m} \tag{3.9}$$

where x_m^0 relates to the space coordinate of the m'th soliton at $t = 0$ in a way that will be clarified later. The phase ψ_m relates to the topological quantum number defined as the difference between the values the affine Toda field takes at large distances. Certain discrete values are forbidden by the requirement that the solution (3.1) should not develop singularities as x varies over space.

These soliton solutions exhibit a number of important features. Despite the imaginary nature of β, the energy and momentum of the solution (3.1) has been evaluated and shown to be real and finite (with positive energy) [9]. Moreover the resulting form is characteristic of N solitons moving with the stated rapidities and masses (3.5). That the affine Toda field interpolates degenerate vacua at large distances can be confirmed explicitly (when $\hat{g}$ is simply laced [13]) using the generalised vertex operator construction [10, 13] which we now explain in more detail.

Consider the single exponential factor (3.3) creating the m'th soliton. Since $\hat{F}^i(\theta)$ is a generator of $\hat{g}$, we must check that the exponential indeed makes sense as a finite operator in representations of the highest weight considered in (3.1). Expanding the exponential as a series, we find that powers of $\hat{F}^i(\theta)$ higher than the level vanish identically if $\hat{g}$ is simply laced [10]. Furthermore, the highest non vanishing power, namely the level, is given by a vertex operator obtained by normal ordering an exponential expression of the principal Heisenberg subalgebra [13]:

$$\frac{\left(\hat{F}^i(\theta)\right)^{m_j}}{(m_j)!} = e^{-2\pi i \lambda_i \cdot \lambda_j} Y_-^i Y_+^i \tag{3.10}$$

where

$$Y^i_\pm = \exp\left\{\sum_{M>0} \frac{\gamma_i \cdot q(\mp[M])z^{\mp M}}{\mp M}\hat{E}_{\pm M}\right\}, \tag{3.11}$$

and z is related to the rapidity θ as in (3.8). It follows that

$$e^{QW_i(\theta)\hat{F}^i(\theta)} = 1 + \ldots + (QW_i(\theta))^{m_j}\, e^{-2\pi i\lambda_i\cdot\lambda_j} Y^i_- Y^i_+. \tag{3.12}$$

The coefficient of the intermediate powers of QW_i are not determined by the argument, even though finite, and so are denoted by the dots in (3.12). It will emerge that the asymptotic properties of solitons that we seek do not depend on these undetermined quantities. If W_i tends to zero, (3.12) is dominated by the first term, unity, and if W_i tends to plus infinity, (3.12) is dominated by the last term, given by the vertex operator.

In particular, for the single soliton solution (3.1) in which $g(t)$ is given by a single factor (3.3), we see that the limits as x tends to $\pm\infty$ are respectively $e^{-2\pi i\lambda_i\cdot\lambda_j}$ and 1. This result assures that the affine Toda field ϕ does interpolate degenerate vacua at $x = \pm\infty$, with the topological charge, $\Delta\phi$ satisfying

$$-\frac{\beta}{2\pi i}\Delta\phi = \lambda_i + \Lambda_R(g), \tag{3.13}$$

where we recall that i labels the relevant soliton species. Similar results apply to solutions describing any number of solitons.

In this paper we shall address more refined questions concerning the asymptotic behaviour of the N-soliton solution (3.1) and in particular determine the lateral displacement of the soliton trajectories arising from the collisions as described in section 2. We shall find that the limited information outlined above is quite sufficient for this purpose, as the unknown coefficients in (3.12) are irrelevant. What is important is the number $X_{ik}(z_i, z_k)$ arising when the product of two of the vertex operators (3.10) is normal ordered [10, 13]

$$Y^i_-(z_i)Y^i_+(z_i)Y^k_-(z_k)Y^k_+(z_k) = (X_{ik}(z_i,z_k))^x\, Y^i_-(z_i)Y^k_-(z_k)Y^i_+(z_i)Y^k_+(z_k) \tag{3.14}$$

where x is the level of the representation considered. It was shown in [10] that

$$X_{ik}(z_i, z_k) = \prod_{p=1}^{h} \left(z_i - e^{\frac{2\pi i p}{h}} z_k \right)^{\gamma_i \cdot \sigma^p \gamma_k}, \qquad |z_i| > |z_k|, \tag{3.15}$$

where the quantities $\sigma, \gamma_i, \gamma_k$ are defined there.

By the commutation relations (3.6) we also find [13]

$$Y_+^i(z_i) \hat{F}^k(z_k) = X_{ik}(z_i, z_k) \hat{F}^k(z_k) Y_+^i(z_i), \tag{3.16}$$

$$\hat{F}^k(z_k) Y_-^i(z_i) = Y_-^i(z_i) \hat{F}^k(z_k) X_{ik}(z_i, z_k). \tag{3.17}$$

We shall show in the next section that the quantity X_{ik} appears in the time delay result (2.10) and that when (3.8) is inserted it enjoys the properties required of this physical interpretation (section 6).

4 Space-time trajectories of two colliding solitons

First we consider solutions with two solitons and want to determine the asymptotic form of their trajectories in space-time and hence the lateral displacements defined in section 2. In the next section we consider collisions of more solitons, finding that the two-soliton result is the fundamental block, as expected in an integrable theory.

The appropriate group element (3.2) contains only two factors

$$g(t) \;=\; e^{Q_2 W_{i(2)}(\theta_2) \hat{F}^{i(2)}(\theta_2)} e^{Q_1 W_{i(1)}(\theta_1) \hat{F}^{i(1)}(\theta_1)}, \tag{4.1}$$

where $\theta_1 > \theta_2$ by (3.8) and (3.15). This inequality means that soliton 1, of species $i(1)$, moves faster than soliton 2, of species $i(2)$. It must therefore start to the left of soliton 2 and eventually overtake it, causing a collision whose outcome we wish to study.

We shall do this by *"tracking"* each soliton in time. By tracking the faster soliton 1 we mean that we hold $W_{i(1)}(\theta_1)$ fixed as time varies. As

$$W_{i(1)}(\theta_1) \;=\; e^{\mu_{i(1)} \cosh\theta_1 (x - v_1 t)}, \tag{4.2}$$

where $v_1 = \tanh\theta_1$ is the velocity of soliton 1, this means that, as t varies, x varies so as to hold $W_{i(1)}$ fixed, thereby remaining in the vicinity of the soliton, which is near $x = v_1 t + x_1^0$. While $W_{i(1)}(\theta_1)$ is held fixed, that is $x - v_1 t$ is fixed, the time dependence of $W_{i(2)}(\theta_2)$ is given by

$$W_{i(2)}(\theta_2) \;=\; e^{\mu_{i(2)}\cosh\theta_2(x-v_2t)} = \text{const}\;\; e^{\mu_{i(2)}\cosh\theta_2(v_1-v_2)t} \quad . \tag{4.3}$$

Hence, in the past, $t \to -\infty$, $W_{i(2)}(\theta_2)$ tends to 0 as $v_1 > v_2$. So, by (3.12)

$$e^{Q_2 W_{i(2)}(\theta_2)\hat{F}^{i(2)}(\theta_2)} \;\to\; 1 \;\; . \tag{4.4}$$

Thus, by (3.1), in the past

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; \frac{\langle\Lambda_j|\; e^{Q_1 W_{i(1)}(\theta_1)\hat{F}^{i(1)}(\theta_1)}\;|\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_1 W_{i(1)}(\theta_1)\hat{F}^{i(1)}(\theta_1)}\;|\Lambda_0\rangle^{m_j}} \tag{4.5}$$

which we recognise as a single soliton solution of species $i(1)$, velocity v_1 and phase ψ_1. For comparision, let us track soliton 1 in the two soliton solution into the future, so $t \to \infty$, with $W_{i(1)}$ fixed, so that $W_{i(2)}$ tends to plus infinity. Now by (3.12) the exponential is dominated by the highest non-vanishing power. In the numerator of (3.10) which has level m_j this yields

$$e^{Q_2 W_{i(2)}(\theta_2)\hat{F}^{i(2)}(\theta_2)} \;\to\; e^{-2\pi i\lambda_{i(2)}\cdot\lambda_j}(Q_2 W_{i(2)})^{m_j} Y_-^{i(2)} Y_+^{i(2)} \quad . \tag{4.6}$$

The factor $Y_-^{i(2)}$ annihilates to unity on the highest weight state $\langle\Lambda_j|$ leaving the factor $Y_+^{i(2)}$, which would likewise annihilate to unity on the right were it not for the intervening factor $e^{Q_1 W_{i(1)}(\theta_1)\hat{F}^{i(1)}(\theta_1)}$. By (3.16) these factors can be interchanged if Q_1 is replaced by $Q_1 X_{i(1)i(2)}(\theta_{12})$. Similar operations can be applied to eliminate the vertex operator from the denominator. The large factors $(Q_2 W_{i(2)})^{m_j}$ cancel between numerator and denominator, leaving in the future, $t \to \infty$,

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; e^{-2\pi i\lambda_{i(1)}\cdot\lambda_j}\frac{\langle\Lambda_j|\; e^{Q_1 W_{i(1)}(\theta_1)X_{i(1)i(2)}(\theta_{12})\hat{F}^{i(1)}(\theta_1)}\;|\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_1 W_{i(1)}(\theta_1)X_{i(1)i(2)}(\theta_{12})\hat{F}^{i(1)}(\theta_1)}\;|\Lambda_0\rangle^{m_j}} \quad . \tag{4.7}$$

Again we recognise a single soliton solution of species $i(1)$, rapidity θ_1 and phase ψ_1. The phase factor preceding (4.7) is innocuous, representing a translation of ϕ

by $\frac{2\pi i}{\beta}\lambda_{i(1)}$, a symmetry of the theory. The other difference between (4.6) and (4.7) is significant. Since $X_{i(1)i(2)}(\theta_{12})$ is real and positive, it means that Q_1 has acquired a factor $X_{i(1)i(2)}(\theta_{12})$ which changes its modulus (but not the phase), and hence x_1^0 (see (3.9)) in the evolution from the past to the future. The effect is that

$$\mu_{i(1)} \cosh\theta_1(x - v_1 t) \to \mu_{i(1)} \cosh\theta_1(x - v_1 t) + \ln X_{i(1)i(2)}(\theta_{12}) \tag{4.8}$$

so that the solution (4.7) differs from (4.6) by a translation in space-time. In particular the trajectories in space-time of the outgoing soliton is translated with respect to the ingoing soliton. Comparing with (2.3) and the subsequent discussion we see

$$E_1\Delta_{12}(x) \;=\; -\frac{M_{i(1)}}{\mu_{i(1)}}\; \ln X_{i(1)i(2)}(\theta_{12}) \tag{4.9}$$

as the energy of the soliton is $E_1 = \mu_{i(1)} \cosh\theta_1$. Employing the mass formula (3.5), this equals

$$E_1\Delta_{12}(x) \;=\; -\;\frac{2h}{|\beta^2|}\ln X_{i(1)i(2)}(\theta_{12}) \tag{4.10}$$

which is the announced result (2.10) for the faster soliton.

Now let us derive the corresponding result for the slower soliton 2, by tracking it. As $W_{i(2)}$ is now held fixed

$$W_{i(1)}(\theta_1) \;=\; \text{const}\;\; e^{\mu_{i(1)}\cosh\theta_1 (v_2 - v_1)t} \quad . \tag{4.11}$$

tends to ∞ and 0 in the past and future, respectively. Thus, in the past,

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; e^{-2\pi i\lambda_j\cdot\lambda_{i(1)}} \frac{\langle\Lambda_j|\; e^{Q_2 W_{i(2)}(\theta_2) X_{i(1)i(2)}(\theta_{12}) \hat{F}^{i(2)}(\theta_2)}\; |\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_2 W_{i(2)}(\theta_2) X_{i(1)i(2)}(\theta_{12}) \hat{F}^{i(2)}(\theta_2)}\; |\Lambda_0\rangle^{m_j}} \quad . \tag{4.12}$$

using (3.16) with (3.17), whereas in the future

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; \frac{\langle\Lambda_j|\; e^{Q_2 W_{i(2)}(\theta_2) \hat{F}^{i(2)}(\theta_2)}\; |\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_2 W_{i(2)}(\theta_2) \hat{F}^{i(2)}(\theta_2)}\; |\Lambda_0\rangle^{m_j}} \quad . \tag{4.13}$$

So $Q_2 X_{i(1)i(2)}(\theta_{12}) \to Q_2$ during the evolution and

$$E_2\Delta_{21}(x) \;=\; +\;\frac{2h}{|\beta^2|}\ln X_{i(1)i(2)}(\theta_{12}), \tag{4.14}$$

thereby confirming (2.10) for the slower soliton.

Notice that the solution considered describes only a transmission and not a reflection of solitons. The only possible exception is when the species $i(1)$ and $i(2)$ coincide as do the phases ψ_1 and ψ_2. Then we cannot tell whether the scattering is transmissive or reflective.

It is interesting to repeat the calculation with the order of the two factors in (4.1) reversed. The reader will find that the asymptotic results (4.5), (4.7), (4.12) and (4.13) are unchanged, as is therefore the spatial displacement.

5 Space-time trajectories of any number of colliding solitons

It is not difficult to extend the preceding argument from the collision of two solitons to the collision of any number of various species. The interesting result is that the total displacement of the space-time trajectories of any chosen soliton is precisely the sum of the contributions previously found for the collision of the chosen soliton with each of the others. This sum is independent of the ordered sequence in which the chosen soliton collides with the others. Hence this result is the classical analogue of the Yang-Baxter [15] and bootstrap relations [16] governing quantum scattering in integrable theories.

The solution (3.1) and (3.2) describes N solitons. We shall choose to track one of these, say soliton m, by remaining close to its trajectory in space-time as the past evolves into the future. So $W_{i(m)}$, (3.4) , is held fixed and the other functions $W_{i(n)}$ behave as

$$W_{i(n)} \; = \; \text{const}\; e^{\mu_{i(n)} \cosh\theta_2 (v_m - v_n)t} \quad . \tag{5.1}$$

Thus if soliton n is slower than soliton m, $W_{i(n)}$ tends to 0 in the past and ∞ in the future. If soliton m is faster than soliton n, then the limits are reversed. Repeating the arguments of the preceding section we find that in the past

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; \exp\left(-2\pi i \sum_{v_k<v_m} \lambda_j\cdot\lambda_{i(k)}\right) \frac{\langle\Lambda_j|\; e^{Q_m(I)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_m(I)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_0\rangle^{m_j}} \tag{5.2}$$

where $Q_m(I) = Q_m \prod_{v_k<v_m} X_{i(m)i(k)}$.

In the future

$$e^{-\beta\lambda_j\cdot\phi} \;\to\; \exp\left(-2\pi i \sum_{v_k>v_m} \lambda_j\cdot\lambda_{i(k)}\right) \frac{\langle\Lambda_j|\; e^{Q_m(F)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_j\rangle}{\langle\Lambda_0|\; e^{Q_m(F)W_{i(m)}(\theta_m)\hat{F}^{i(m)}(\theta_m)}\;|\Lambda_0\rangle^{m_j}} \tag{5.3}$$

where $Q_m(F) = Q_m \prod_{v_k>v_m} X_{i(m)i(k)}$. This immediately yields the announced results:

$$E_m\Delta_m(x) = -p_m\Delta_m(t) = \frac{2h}{|\beta^2|}\left(\sum_{v_k>v_m} \ln X_{i(m)i(k)} - \sum_{v_k<v_m} \ln X_{i(m)i(k)}\right) \tag{5.4}$$

where $\Delta_m(x), \Delta_m(t)$ denote the displacement (2.3) and (2.5) for the m^{th} soliton.

Notice that this result (5.4) does not depend on the values of the $Q_{i(n)}$, but only on the rapidities of the solitons. Hence the temporal order of the scattering can be altered without changing the rapidities of the solitons. Thus the overall displacement of the trajectories of soliton m, (5.4) is independent of the order in which the collisions occurred.

The same procedure can be applied to each of the other $(N-1)$ solitons. In this way we see how the N soliton solution asymptotically contains the N single soliton solutions in both the past and future. The only alterations in time are the lateral displacement of the trajectories specified by our result (5.4).

6 Properties of the function $X_{ik}(\theta)$

The factor

$$X_{ik}(z_i, z_k) = \prod_{p=1}^{h} \left(z_i - e^{\frac{2\pi i p}{h}} z_k\right)^{\gamma_i\cdot\sigma^p\gamma_k} \tag{6.1}$$

arose [10, 13] in the normal ordering of the product of the two vertex operators, (3.14),

$$\frac{(\hat{F}^i(z_i))^{m_j}}{m_j!} \quad \text{and} \quad \frac{(\hat{F}^k(z_k))^{m_j}}{m_j!}$$

The exponents, $\gamma_i \cdot \sigma^p \gamma_k$, being scalar products of roots of a simply laced Lie algebra, can only take the values $0, \pm 1, \pm 2$. Thus $X_{ik}(z_i, z_k)$ can be analytically extended to a meromorphic function of the complex variables z_i and z_k. X_{ik} only possesses a double pole if $i = \bar{k}$, while the occurrence of simple poles is governed by Dorey's fusing rule [23, 10, 24].

Using the first, (6.2), of the two facts that

$$\sum_{p=1}^{h} \sigma^p \gamma_k = 0 \tag{6.2}$$

and

$$\sum_{p=1}^{h} p \gamma_i \cdot \sigma^p \gamma_k \in h\mathbb{Z}, \tag{6.3}$$

we can rewrite (6.1) as

$$\prod_{p=1}^{h} (z_i z_k^{-1} - e^{\frac{2\pi i p}{h}})^{\gamma_i \cdot \sigma^p \gamma_k} \tag{6.4}$$

which means that $X_{ik}(z_i, z_k)$ depends on z_i and z_k only through the ratio $z_i z_k^{-1}$. Furthermore using both (6.2) and (6.3) we find that it exhibits the symmetry property

$$X_{ik}(z_i, z_k) = X_{ki}(z_k, z_i), \tag{6.5}$$

which means that the vertex operators (3.10) braid trivially. This appears to be the explanation of our earlier observations that the order of the soliton factors in (3.2) is irrelevant. Introducing the soliton rapidity θ_k via (3.8),

$$z_k = i e^{-\theta_k} e^{-\frac{i\pi}{2h}(1+c(k))}$$

we find that X_{ik} can be expressed as a function of the rapidity difference $\theta = \theta_i - \theta_k$:

$$X_{ik}(z_i, z_k) = \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k} = X_{ik}(\theta). \tag{6.6}$$

Thus $X_{ik}(\theta)$ is Lorentz invariant since the relative rapidity is. This is in accord with our result (2.10).

The result (2.10) which was established in the preceding sections means that $X_{ik}(\theta)$ has a space-time interpretation in terms of the scattering of solitons. As a consequence it ought to be a real number (when θ is real) and exhibit some further symmetry properties.

We shall now check these properties explicitly, showing that $X_{ik}(\theta)$ has period $2\pi i$,

$$X_{ik}(\theta + 2\pi i) = X_{ik}(\theta), \tag{6.7}$$

is symmetric in the sense

$$X_{ik}(\theta) = X_{ki}(\theta), \tag{6.8}$$

is even in θ

$$X_{ik}(\theta) = X_{ik}(-\theta), \tag{6.9}$$

takes values in the unit interval when θ is real

$$0 \le X_{ik}(\theta) < 1, \qquad \theta \in I\!R, \tag{6.10}$$

and obeys the "crossing" property

$$X_{\bar{\imath}k}(\theta) = (X_{ik}(\theta + i\pi))^{-1} \tag{6.11}$$

where $\bar{\imath}$ denotes the anti-species of i.

Notice that the periodic property (6.7) is already evident from (6.6). The symmetry property (6.8) follows using the identity

$$\gamma_i \cdot \sigma^p \gamma_k = \gamma_k \cdot \sigma^{p'} \gamma_i,$$

where

$$2p + \frac{c(i) - c(k)}{2} = 2p' + \frac{c(k) - c(i)}{2}.$$

To prove the evenness property (6.9) note that

$$\begin{aligned} X_{ik}(-\theta) &= \prod_{p=1}^{h} \left(e^{\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k} \\ &= \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{-\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k} \end{aligned}$$

by (6.2) and (6.3). Now use $\gamma_i \cdot \sigma^p \gamma_k = \gamma_k \cdot \sigma^{-p} \gamma_i$ to recognize, on changing the dummy label $p \to -p$, $X_{ki}(\theta)$ which equals $X_{ik}(\theta)$ by the symmetry property (6.8). To prove the reality property first note that

$$X_{ik}(\theta^*)^* = \prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{-\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{\gamma_i \cdot \sigma^p \gamma_k},$$

which equals $X_{ik}(-\theta)$ by (6.2) and (6.3) and hence $X_{ik}(\theta)$ by evenness. Thus $X_{ik}(\theta)$ is real when θ is.

Now let us consider the possibility that $X_{ik}(\theta)$ has zeroes or poles when θ is real. This is only possible when a factor vanishes, so that both

$$\theta = 0, \qquad \text{and} \quad p + \frac{1}{4}(c(i) - c(k)) = 0 \quad \text{mod } h.$$

The second condition implies that $c(i) = c(k)$ and that $p = h$. When $c(i) = c(k)$, $\gamma_i \cdot \sigma^h \gamma_k = \gamma_i \cdot \gamma_k$ vanishes unless $i = k$ when it equals 2. So for real θ, $X_{ik}(\theta)$ has no poles and the only zero occurs when $i = k$ and $\theta = 0$. We already knew that $X_{ii}(0)$ had to be zero as it implies the nilpotency condition $(\hat{F}^i(\theta))^2 = 0$ at level 1.

Now let us prove that $X_{ik}(\theta)$ takes values in the unit interval. The argument is intriguingly similar to that of section (4.5) of [18] concerned with positivity properties of the affine Toda particle scattering matrix.

Using the relation

$$\gamma_i \cdot \sigma^p \gamma_k = \lambda_i \cdot \sigma^{-p + \frac{c(k)-1}{2}} \gamma_k - \lambda_i \cdot \sigma^{-p + \frac{c(k)+1}{2}} \gamma_k,$$

we can rewrite (6.6) as

$$\begin{aligned} X_{ik}(\theta) &= \frac{\prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{-\lambda_i \cdot \sigma^{-p + \frac{c(k)+1}{2}} \gamma_k}}{\prod_{p=1}^{h} \left(e^{-\theta} - e^{\frac{\pi i}{h}(2p + \frac{c(i)-c(k)}{2})} \right)^{-\lambda_i \cdot \sigma^{-p + \frac{c(k)-1}{2}} \gamma_k}} \\ &= \prod_{p=1}^{h} \left[\frac{\sinh \frac{1}{2}(\theta - \frac{\pi i}{h}(2p - \frac{c(i)+c(k)}{2} - 1))}{\sinh \frac{1}{2}(\theta - \frac{\pi i}{h}(2p - \frac{c(i)+c(k)}{2} + 1))} \right]^{-\lambda_i \cdot \sigma^p \gamma_k} \end{aligned}$$

on relabelling the dummy index in order to gather the factors under a common exponent. The factors can be further paired using the fact [18] that

$$\lambda_i \cdot \sigma^p \gamma_k = -\lambda_i \cdot \sigma^{p'} \gamma_k,$$

where

$$p' = h + \frac{c(i) + c(k)}{2} - p.$$

Using this we can rewrite $X_{ik}(\theta)$ as

$$X_{ik}(\theta) = \prod_{p=a}^{b} \left[\frac{\cosh(\theta) - \cos\frac{\pi}{h}(2p - \frac{c(i)+c(k)}{2} - 1)}{\cosh(\theta) - \cos\frac{\pi}{h}(2p - \frac{c(i)+c(k)}{2} + 1)} \right]^{-\lambda_i \cdot \sigma^p \gamma_k} \tag{6.12}$$

where $a = \frac{1+c(k)}{2}$ and $b = \frac{h-1}{2} + \frac{c(k)+c(\bar{k})}{4}$. The significance of the reduced range of p is that, in it,

$$\lambda_i \cdot \sigma^p \gamma_k \leq 0.$$

Thus all the exponents in (6.12) are positive. Thus in order to prove (6.10) it would be sufficient to show that each factor in (6.12) individually lies between 0 and 1. Because $\cosh\theta \geq 1$ this is ensured provided

$$1 \geq \cos\frac{\pi}{h}\left(2p - \frac{c(i) + c(k)}{2} - 1\right) > \cos\frac{\pi}{h}\left(2p - \frac{c(i) + c(k)}{2} + 1\right) > -1, \tag{6.13}$$

which follows from the fact that $\cos\phi$ is monotonically decreasing from 1 to -1 in the interval $0 < \phi < \pi$, providing

$$0 \leq 2p - \frac{c(i) + c(k)}{2} - 1 < 2p - \frac{c(i) + c(k)}{2} + 1 < h.$$

The smallest value of $2p - (c(i) + c(k))/2 - 1$ occurs when $p = a = (1 + c(k))/2$ which is $(c(k) - c(i))/2$. This can only be negative when $c(k) = -1 = -c(i)$, so that $p = 0$. In this case the exponent $\lambda_i \cdot \sigma^p \gamma_k = \lambda_i \cdot \gamma_k = 0$, as $i \neq k$, and the factor contributes unity. A similar discussion applies to the upper limit b. Notice that expression (6.12) is explicitly real and even in θ.

Finally, using the relation [18],

$$\gamma_i = -\sigma^{-\frac{h}{2} - \frac{c(i) - c(\bar{\imath})}{4}} \gamma_{\bar{\imath}},$$

we obtain the "crossing" property (6.11)

$$\begin{aligned} X_{ik}(\theta+i\pi) &= \prod_{p=1}^{h}\left(e^{-\theta}-e^{\frac{\pi i}{h}(2p-h+\frac{c(i)-c(k)}{2})}\right)^{\gamma_i\cdot\sigma^p\gamma_k} \\ &= \prod_{p=1}^{h}\left(e^{-\theta}-e^{\frac{\pi i}{h}(2p+\frac{c(\bar{\imath})-c(i)}{2})}\right)^{\gamma_{\bar{\imath}}\cdot\sigma^p\gamma_k} \\ &= (X_{\bar{\imath}k}(\theta))^{-1}. \end{aligned}$$

Notice that, in agreement with the results of [13], the crossing property involves the analytic continuation $\theta \to \theta + i\pi$ rather than $\theta \to i\pi - \theta$ for the reasons explained by Coleman [25], namely that the semiclassical approximation breaks down on the imaginary rapidity axis.

It is further worth noting that by similar manipulations one can show that $X_{ij}(\theta)$ additionally satisfies the bootstrap equation [17, 18] and thereby enhances the remarkable similarity in structure between $X_{ij}(\theta)$ and the scattering matrix.

7 Conclusions

There are two main conclusions to our work and a number of comments. The first result we have established is the intimate connection between the space-time properties of the affine Toda solitons and the vertex operators associated with them through the numerical function $X_{ik}(\theta)$ arising when the product of the two vertex operators is normal ordered. This connection is remarkable in view of the fact that these vertex operators do not provide complete information concerning the solitons as they do not completely determine the Kac-Moody group element (3.3) but merely yield (3.12).

There is a well known result of Eisenbud and Wigner [26, 27] relating the time delay to the quantum mechanical scattering matrix in the semi-classical approximation. The phase shift is obtained by integrating the time delay with respect to energy, introducing a constant of integration proportional to the number of bound states, presumably breathers in our context. This result has been exploited in sine-Gordon theory [21, 11] but the breather spectrum for general affine Toda field theory requires further study.

These results would presumably shed light on the intriguing similarity in structure between $X_{ik}(\theta)$ and properties of particle scattering matrix elements in affine Toda field theories [17, 18] as well as the ideas of Corrigan and Dorey [28] for obtaining the S-matrices from the braiding of vertex operators representing the Faddeev-Zamolodchikov operators [16, 29] .

Our second main result is that, since $X_{ik}(\theta)$ takes values between 0 and 1, it follows that the time delay experienced by any soliton in collision with any other in their centre of momentum frame is negative. This strongly suggests that the forces between any two solitons is always attractive. This further suggests that bound states (breathers) will form, though as far as we know, this can only occur when the two solitons have equal mass and are anti-species of each other. The only exception to the statement that forces are attractive is when the two solitons involved are indistinguishable. Then we can no longer recognize from the explicit solution that the scattering is only a transmission as it could be reflective in this case. If the scattering is considered to be reflective, the time advance can be ascribed to a repulsive core. This is the accepted picture in sine-Gordon theory [21] where independent arguments imply that the forces between indistinguishable solitons are repulsive.

We should like to mention a delicate point here. We can distinguish the outgoing solitons if they have different species or, if not, possess different phases in Q. It is believed that the phase in Q, (3.9), is related to the topological quantum number of the soliton, (3.13). Unfortunately this connection has not been established in a satisfactorily general way and the correspondence is not one to one. As mentioned above, certain discrete phases are forbidden in order that the soliton solutions be nonsingular. The danger concerns possible zeros of the expectation values of $g(t)$, (3.1), or τ-functions, as x varies over space. This leaves disconnected allowed ranges for the phase which seem to correspond to specific values of the topological quantum number (3.13). The topological quantum number automatically takes discrete values and is a continuous function of Q except for discontinuities occurring across the forbidden boundaries but the exact details are only understood in the $su(n)$ case considered by McGhee [30]. This remains an outstanding issue. We should like to be able to say that two solitons of the same species are distinguishable only if they carry different topological quantum numbers and not just different phases, so two solitons could be indistinguishable if they have the same topological quantum number but different phases but this is not yet understood.

A second intriguing point concerns the repulsive core just mentioned for indistinguishable solitons. This partly tallies with the fact that it is impossible for two solitons of the same species to have the same rapidities. This is because

$$\lim_{\theta_1,\theta_2\to\theta} e^{Q_1 W_1 \hat{F}^i(\theta_1)} e^{Q_2 W_2 \hat{F}^i(\theta_2)} = e^{(Q_1+Q_2) W \hat{F}^i(\theta)}$$

which creates a single soliton rather than two. This phenomenon is familiar in sine-Gordon theory where it is well known that, in the quantum theory, the solitons are the fermions in the massive Thirring model [31, 32, 33]. It appears that something like the exclusion principle is operating at the classical level. The two results, repulsive core and exclusion principle, suggest that the affine Toda solitons may also have a fermionic nature, but again much more needs to be understood.

Finally we mention further remaining questions such as the extension to non simply laced theories, and the question of time delays for the scattering of breathers, once their spectrum is understood.

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