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Form Factors for Integrable Lagrangian Field Theories, the Sinh-Gordon Model

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Abstract

Using Watson's and the recursive equations satisfied by matrix elements of local operators in two-dimensional integrable models, we compute the form factors of the elementary field $\phi(x)$ and the stress-energy tensor $T_{\mu\nu}(x)$ of Sinh-Gordon theory. Form factors of operators with higher spin or with different asymptotic behaviour can easily be deduced from them. The value of the correlation functions are saturated by the form factors with lowest number of particle terms. This is illustrated by an application of the form factors of the trace of $T_{\mu\nu}(x)$ to the sum rule of the c-theorem.

1 Introduction

Recent investigations on two-dimensional quantum field theories have established the exact integrability for a variety of physically interesting models with massive excitations. A rather simple characterization of such theories may be given in terms of their scattering data, i.e. their properties on mass-shell. In fact, the existence of an infinite number of commuting conserved charges implies that the scattering processes which occur in these theories preserve the number of particles and the set of their asymptotic momenta [1]. The computation of the exact factorized S-matrix may be performed by combining the standard requirements of unitarity and crossing symmetry together with the symmetry properties of the model [1-7]. In many cases, the conjectured S-matrix may be supported by perturbative checks [1, 6, 7, 8].

The knowledge of the exact S-matrix can then be used to compute off-shell quantities, like correlation functions of elementary or composite fields of the integrable models under investigation. This can be achieved by considering the form factors of local fields, which are matrix elements of operators between asymptotic states. General properties of unitarity, analyticity and locality lead to a system of functional equations for these matrix elements which permit in many cases their explicit determination [10-17]. The correlation functions are then written in terms of an infinite sum over the multi-particle form factors.

In this paper we investigate one of the simplest integrable Lagrangian system, namely the Sinh-Gordon theory. Many properties of this model are well established, the exact S-matrix for instance was obtained in [6], whereas the quantization of this theory has been studied in [18]. Our objective is to derive expressions for the form factors of the elementary field $\phi(x)$ and the energy-momentum tensor $T_{\mu\nu}(x)$, which are the most representative operators of the odd and even sector of the Z_2 symmetry of this model.

The paper is organized as follows: in section 2 we discuss general properties of form factors for integrable models, i.e. their analytic structure, Watson's and the recursive equations which they satisfy. In section 3 we recall the basic properties of the Sinh-Gordon theory. Section 4 is devoted to the explicit computation of form factors for this theory. In section 5 we investigate the natural grading introduced in the space of the form factors by the arbitrariness inherent Watson's equations and, in particular, we show how form factors of operators with higher spin can be obtained from the ones for $\phi(x)$ and $T_{\mu\nu}(x)$. In section 6 we make use of the form factors of the stress-energy tensor in order to illustrate the c-theorem. In the last section there are our conclusions.

2 General Properties of Form Factors

Essential input for the computation of form factors is the knowledge of the scattering matrix S. For two-dimensional integrable systems the expression of the S-matrix is particularly simple and may be obtained explicitly for several systems [1-7]. Since the dynamics is governed by an infinite number of higher conservation laws, the scattering processes for integrable models are purely elastic and the general n-particle S-matrix defined by

$$S_n(p_1,\dots,p_n) \;=\; {}_{\rm out}\!<p_1,\dots,p_n \mid p_1,\dots,p_n>_{\rm in}\;, \tag{2.1}$$

factorizes into $n(n-1)/2$ two-particle S-matrices [1]

$$S^{(n)}(p_1,p_2,\dots,p_n) \;=\; \prod_{i<j} S_{ij}^{(2)}(p_i,p_j) \quad . \tag{2.2}$$

It is convenient to use instead of the momenta the rapidities β_i, defined by

$$p_i^0 \;=\; m_i \cosh\,\beta_i \;\;,\quad p_i^1 \;=\; m_i \sinh\,\beta_i \;\;. \tag{2.3}$$

By Lorentz invariance, the scattering amplitudes will be functions of the rapidity differences $\beta_{ij} = \beta_i - \beta_j$. The two-particle S-matrix satisfies the usual axioms of

unitarity and crossing symmetry

$$\begin{aligned} S_{ij}(\beta_{ij}) &= S_{ji}(\beta_{ij}) = S_{ij}^{-1}(-\beta_{ij}) \ , \\ S_{i\bar{j}}(\beta_{ij}) &= S_{ij}(i\pi - \beta_{ij}) \ . \end{aligned} \tag{2.4}$$

Possible bound states will occur as simple or higher odd poles in the S-matrix for purely imaginary values of β in the physical strip $0 < \mathrm{Im}\beta < \pi$.

Once the S-matrix is known, it is possible to analyze the off-shell quantum field theory by considering the form-factors which are matrix elements of local operators $\mathcal{O}(x)$ between the asymptotic states. Pioneering work on this subject has been carried out by the authors of ref. [10] and more recently advances have been made by Smirnov and Kirillov [11, 12, 13]. In order to provide a self-consistent account of the paper, we recall some essential properties of the form-factors, staying close to the notations of ref. [11].

2.1 Zamolodchikov algebra

At the heart of the construction of the form factors lies the assumption that there exists a set of vertex operators, of creation and annihilation type, i.e. $V_{\alpha_i}^{\dagger}(\beta_i)$, $V_{\alpha_i}(\beta_i)$, which provide a generalization of the bosonic and fermionic algebras. Here the α_i denotes some quantum number indicating the different types of particles present in the theory. These operators are assumed to obey the following non-Abelian, associative algebra, involving the S-matrix

$$V_{\alpha_i}(\beta_i)V_{\alpha_j}(\beta_j) = S_{ij}(\beta_{ij})\, V_{\alpha_j}(\beta_j)V_{\alpha_i}(\beta_i) \tag{2.5}$$

$$V_{\alpha_i}^{\dagger}(\beta_i)V_{\alpha_j}^{\dagger}(\beta_j) = S_{ij}(\beta_{ij})\, V_{\alpha_j}^{\dagger}(\beta_j)V_{\alpha_i}^{\dagger}(\beta_i) \tag{2.6}$$

$$V_{\alpha_i}(\beta_i)V_{\alpha_j}^{\dagger}(\beta_j) = S_{ij}(\beta_{ji})\, V_{\alpha_j}^{\dagger}(\beta_j)V_{\alpha_i}(\beta_i) + 2\pi\delta_{\alpha_i\alpha_j}\delta(\beta_{ij})\,. \tag{2.7}$$

Each commutation of these operators is thus interpreted as a scattering process. The Poincaré group generated by the Lorentz transformation $L(\epsilon)$ and the translation

T_y is expected to act on these operators in the following way

$$U_L V_\alpha(\beta) U_L^{-1} = V_\alpha(\beta+\epsilon) \tag{2.8}$$

$$U_{T_y} V_\alpha(\beta) U_{T_y}^{-1} = e^{ip_\mu(\beta)y^\mu} V_\alpha(\beta). \tag{2.9}$$

Clearly, the explicit form of these vertex operators depends crucially on the nature of the theory under consideration and a realization of such a construction remains hitherto an open challenge for most theories.

2.2 Physical States

We can use the vertex operators introduced in the previous section in order to define a space of physical states. For this aim, let us consider the vacuum $|0\rangle$ which is the state annihilated by the operator $V_\alpha(\beta)$,

$$V_\alpha(\beta)|0\rangle = 0 = \langle 0|V_\alpha^\dagger(\beta). \tag{2.10}$$

The Hilbert space is then defined by a successive action of $V_\alpha^\dagger(\beta)$ on $|0\rangle$, *i.e.*

$$|V_{\alpha_1}(\beta_1)\dots V_{\alpha_n}(\beta_n)\rangle \equiv V_{\alpha_1}^\dagger(\beta_1)\dots V_{\alpha_n}^\dagger(\beta_n)|0\rangle \ . \tag{2.11}$$

From equation (2.7), the one particle states are normalized as

$$\langle V_{\alpha_i}(\beta_i)|V_{\alpha_j}(\beta_j)\rangle = 2\pi\,\delta_{\alpha_i\alpha_j}\delta(\beta_{ij})\,. \tag{2.12}$$

The algebra of the vertex operators implies that the vectors (2.11) are not linearly independent and in order to obtain a basis of linearly independent states we require some additional restrictions. In [1] the following prescription was proposed: Select as a basis for the in-states those which are ordered with decreasing rapidities

$$\beta_1 > \dots > \beta_n$$

and as a basis for the out-states those with increasing rapidities

$$\beta_1 < \dots < \beta_n \ .$$

These conditions select a set of linearly independent vectors which serve as a unique basis.

2.3 Form Factors

If not explicitly mentioned, in the following we will consider matrix elements between in-states and out-states of hermitian local scalar operators $\mathcal{O}(x)$ of a theory with only one self-conjugate particle

$$ {}_{\text{out}}\langle V(\beta_{m+1})\dots V(\beta_n)|\mathcal{O}(x)|V(\beta_1)\dots V(\beta_m)\rangle_{\text{in}}. \tag{2.13} $$

Matrix elements of higher spin operators will be discussed later on. We can always shift the matrix elements (2.13) to the origin by means of a translation on the operator $\mathcal{O}(x)$, i.e. $U_{T_y}\mathcal{O}(x)U_{T_y}^{-1} = \mathcal{O}(x+y)$ and by using eq. (2.9),

$$ \begin{aligned} &\exp\left[i\left(\sum_{i=m+1}^{n} p_\mu(\beta_i) - \sum_{i=1}^{m} p_\mu(\beta_i)\right)x^\mu\right] \\ &\times {}_{\text{out}}\langle V(\beta_{m+1})\dots V(\beta_n)|\mathcal{O}(0)|V(\beta_1)\dots V(\beta_m)\rangle_{\text{in}} \ . \end{aligned} \tag{2.14} $$

It is convenient to introduce the following functions, called form factors (fig. 1)

$$ F_n^{\mathcal{O}}(\beta_1,\beta_2,\dots,\beta_n) \ = \ \langle 0 \mid \mathcal{O}(0,0) \mid \beta_1,\beta_2,\dots,\beta_n\rangle_{in}. \tag{2.15} $$

which are the matrix elements of an operator at the origin between an n-particle in-state and the vacuum*. For local scalar operators $\mathcal{O}(x)$, relativistic invariance implies that the form factors F_n are functions of the difference of the rapidities β_{ij}

$$ F_n(\beta_1,\beta_2,\dots,\beta_n) \ = \ F_n(\beta_{12},\beta_{13},\dots,\beta_{ij},\dots) \quad , i<j \ . \tag{2.16} $$

Crossing symmetry also implies that the most general matrix element (2.14) is obtained by an analytic continuation of (2.15), and equals

$$ F_{n+m}(\beta_1,\beta_2,\dots,\beta_m,\beta_{m+1}-i\pi,\dots,\beta_n-i\pi) \ = \ F_{n+m}(\beta_{ij},i\pi-\beta_{sr},\beta_{kl}) \tag{2.17} $$

where $1\le i<j\le m$, $1\le r\le m<s\le n$, and $m<k<l\le n$.

* Here and in the following we use a more simplified notation for the physical states $|\dots V_{\alpha_n'}(\beta_n')\dots\rangle \equiv |\dots\beta_n\dots\rangle$. In most cases we will also suppress the superscript $\mathcal{O}$ and only use it when considering form factors related to different local operators.

Except for the poles corresponding to the one-particle bound states in all subchannels, we expect the form factors F_n to be analytic inside the strip $0 < \mathrm{Im}\beta_{ij} < 2\pi$.

2.4 Watson's Equations

The form factors of a hermitian local scalar operator $\mathcal{O}(x)$ satisfy a set of equations, known as Watson's equations [9], which for integrable systems assume a particularly simple form

$$\begin{aligned} F_n(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) &= F_n(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n)S(\beta_i-\beta_{i+1}) \ , \qquad (2.18)\\ F_n(\beta_1+2\pi i,\ldots,\beta_{n-1},\beta_n) &= F_n(\beta_2,\ldots,\beta_n,\beta_1) = \prod_{i=2}^{n} S(\beta_i-\beta_1)F_n(\beta_1,\ldots,\beta_n) \ . \end{aligned}$$

The first equation is simply a consequence of (2.5), i.e. as a result of the commutation of two operators we get a scattering process. Concerning the second equation, it states which is the discontinuity on the cuts $\beta_{1i} = 2\pi i$. In the case $n = 2$, eqs. (2.18) reduce to

$$\begin{aligned} F_2(\beta) &= F_2(-\beta)S_2(\beta) \ , \\ F_2(i\pi-\beta) &= F_2(i\pi+\beta) \ . \end{aligned} \qquad (2.19)$$

Smirnov [11, 13] has shown that eqs. (2.18), together with eqs. (2.25) and (2.27) which will be discussed in the next section, can be regarded as a system of axioms which defines the whole local operator content of the theory.

The general solution of Watson's equations can always be brought into the form [10]

$$F_n(\beta_1,\ldots,\beta_n) = K_n(\beta_1,\ldots,\beta_n)\prod_{i<j} F_{\min}(\beta_{ij}) \ , \qquad (2.20)$$

where $F_{\min}(\beta)$ has the properties that it satisfies (2.19), is analytic in $0 \le \mathrm{Im}\ \beta \le \pi$, has no zeros in $0 < \mathrm{Im}\ \beta < \pi$, and converges to a constant value for large values of β. These requirements uniquely determine this function, up to a normalization. The remaining factors K_n then satisfy Watson's equations with $S_2 = 1$, which implies

that they are completely symmetric, $2\pi i$-periodic functions of the β_i. They must contain all the physical poles expected in the form factor under consideration and must satisfy a correct asymptotic behaviour for large value of β_i. Both requirements depend on the nature of the theory and on the operator $\mathcal{O}$.

Postponing the discussion on the pole structure of F_n to the next section, let us notice that one condition on the asymptotic behaviour of the form factors is dictated by relativistic invariance. In fact, a simultaneous shift in the rapidity variables results in

$$F_n^{\mathcal{O}}(\beta_1 + \Lambda, \beta_2 + \Lambda, \ldots, \beta_n + \Lambda) \;=\; F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) \;, \tag{2.21}$$

For form factors of an operator $\mathcal{O}(x)$ of spin s, the previous equation generalizes to

$$F_n^{\mathcal{O}}(\beta_1 + \Lambda, \beta_2 + \Lambda, \ldots, \beta_n + \Lambda) \;=\; e^{s\Lambda}\, F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) \;, \tag{2.22}$$

Secondly, in order to have a power-law bounded ultraviolet behaviour of the two-point function of the operator $\mathcal{O}(x)$ (which is the case we will consider), we have to require that the form factors behave asymptotically at most as $\exp(k\beta_i)$ in the limit $\beta_i \to \infty$, with k being a constant independent of i. This means that, once we extract from K_n the denominator which gives rise to the poles, the remaining part has to be a symmetric function of the variables $x_i \equiv e^{\beta_i}$, with a finite number of terms, i.e. a symmetric polynomial in the x_i's. It is convenient to introduce a basis in this functional space given by the *elementary symmetric polynomials* $\sigma_k^{(n)}(x_1, \ldots, x_n)$ which are generated by [25]

$$\prod_{i=1}^{n}(x + x_i) \;=\; \sum_{k=0}^{n} x^{n-k}\, \sigma_k^{(n)}(x_1, x_2, \ldots, x_n). \tag{2.23}$$

Conventionally the $\sigma_k^{(n)}$ with $k > n$ and with $n < 0$ are zero. The explicit expressions

for the other cases are

$$\begin{aligned}
&\sigma_0 = 1 \ , \\
&\sigma_1 = x_1 + x_2 + \ldots + x_n \ , \\
&\sigma_2 = x_1x_2 + x_1x_3 + \ldots x_{n-1}x_n \ , \\
&\vdots \qquad\qquad \vdots \\
&\sigma_n = x_1x_2 \ldots x_n \ .
\end{aligned} \tag{2.24}$$

The $\sigma_k^{(n)}$ are homogeneous polynomials in x_i of total degree k and of degree one in each variable.

2.5 Pole Structure and Residue Equations for the Form Factors

The pole structure of the form factors induces a set of recursive equations for the F_n which are of fundamental importance for their explicit determination. As function of the rapidity differences β_{ij}, the form factors F_n possess two kinds of simple poles.

The first kind of singularities (which do not depend on whether or not the model possesses bound states) arises from kinematical poles located at $\beta_{ij} = i\pi$. They are related to the one-particle pole in a subchannel of three-particle states which, in turn, corresponds to a crossing process of the elastic S-matrix. The corresponding residues are computed by the LSZ reduction [12, 13] and give rise to a recursive equation between the n-particle and the $(n+2)$-particle form factors (fig. 2)

$$-i \lim_{\tilde{\beta}\to\beta}(\tilde{\beta}-\beta)F_{n+2}(\tilde{\beta}+i\pi,\beta,\beta_1,\beta_2,\ldots,\beta_n) = \left(1-\prod_{i=1}^{n} S(\beta-\beta_i)\right) F_n(\beta_1,\ldots,\beta_n). \tag{2.25}$$

The second type of poles in the F_n only arise when bound states are present in the model. These poles are located at the values of β_{ij} in the physical strip which correspond to the resonance angles. Let $\beta_{ij} = iu_{ij}^k$ be one of such poles associated

to the bound state A_k in the channel $A_i \times A_j$. For the S-matrix we have (fig. 3)

$$-\,i \lim_{\beta \to i u_{ij}^k} (\beta - i u_{ij}^k)\, S_{ij}(\beta) \;=\; \left(\Gamma_{ij}^k\right)^2 \tag{2.26}$$

where Γ_{ij}^k is the three-particle vertex on mass-shell. The corresponding residue for the F_n is given by [12, 13]

$$-\,i \lim_{\epsilon \to 0} \epsilon\, F_{n+1}(\beta + i\overline{u}_{ik}^j - \epsilon, \beta - i\overline{u}_{jk}^i + \epsilon, \beta_1, \ldots, \beta_{n-1}) \;=\; \Gamma_{ij}^k\, F_n(\beta, \beta_1, \ldots, \beta_{n-1}) \;\;, \tag{2.27}$$

where $\overline{u}_{ab}^c \equiv (\pi - u_{ab}^c)$. This equation establishes a recursive structure between the $(n+1)$- and n-particle form factors (fig. 4).

2.6 Correlation Functions from Form Factors

Once the form factors of a theory are known, the correlation functions of local operators can be written as an infinite series over multi-particle intermediate states. For instance, the two-point function of an operator $\mathcal{O}(x)$ in real Euclidean space is given by

$$\begin{aligned} \langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle \;&= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} < 0|\mathcal{O}(x)|\beta_1, \ldots, \beta_n >_{\text{in in}} < \beta_1, \ldots, \beta_n|\mathcal{O}(0)|0 > \\ &= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} \mid F_n(\beta_1 \ldots \beta_n) \mid^2 \exp\left(-mr \sum_{i=1}^{n} \cosh \beta_i\right) \end{aligned} \tag{2.28}$$

where r denotes the radial distance, i.e. $r = \sqrt{x_0^2 + x_1^2}$. All integrals are convergent and one expects a convergent series as well. Similar expressions can be derived for multi-point correlators.

3 The Sinh-Gordon Theory

In this paper the model we are concerned with is the Sinh-Gordon theory, defined by the action

$$\mathcal{S} \;=\; \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{g^2} \cosh\, g\phi(x)\right] \;. \tag{3.1}$$

It is the simplest example of an affine Toda Field Theories [19], possessing a Z_2 symmetry $\phi \to -\phi$. By an analytic continuation in g, i.e $g \to ig$, it can formally be mapped to the Sine-Gordon model.

There are numerous alternative viewpoints for the Sinh-Gordon model. First, it can be regarded either as a perturbation of the free massless conformal action by means of the relevant operator[†] $\cosh\, g\phi(x)$. Alternatively, it can be considered as a perturbation of the conformal Liouville action

$$\mathcal{S} \,=\, \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \lambda e^{g\phi}\right] \tag{3.2}$$

by means of the relevant operator $e^{-g\phi}$ or as a conformal affine A_1-Toda Theory [20] in which the conformal symmetry is broken by setting the free field to zero.

Furthermore, it is interesting to notice that the Sinh-Gordon model can be mapped into a Coulomb Gas system with an integer set of charges. To illustrate this, let us consider the (Euclidean) partition function of the model

$$Z(m,g) \,=\, \int \mathcal{D}\phi\, e^{-\mathcal{S}} \quad . \tag{3.3}$$

Using the identity

$$\exp\left(-\frac{m^2}{g^2}\cosh\, g\phi(x)\right) \,=\, \sum_{n(x)=-\infty}^{+\infty} I_{n(x)}\left(-\frac{m^2}{g^2}\right) \exp\left(g\, n(x)\phi(x)\right) \quad , \tag{3.4}$$

where $I_n(a)$ denotes the Bessel function of integer order n, the functional integral in (3.3) becomes Gaussian and can be performed explicitly. Hence $Z(m,g)$ can be cast in the following form

$$Z(m,g) \,=\, Z(0) \sum_{n(x)=-\infty}^{+\infty} I_{n(x)}\left(-\frac{m^2}{g^2}\right) \,\exp\left[\frac{g^2}{2}\int dxdy\, n(x)\Delta(x-y)\, n(y)\right] \quad , \tag{3.5}$$

[†]Although the anomalous dimension of this operator (computed with respect to the free conformal point), is negative, $\Delta = -g^2/8\pi$, the resulting theory is unitary. This is due to the existence of non a nonzero vacuum expectation values of some of the fields $\mathcal{O}_i$ in the theory. A detailed discussion of this point can be found in [15].

where $Z(0)$ is the partition function of a massless free theory and $\Delta(x-y)$ is the two-dimensional massless propagator. The model is therefore equivalent to a Coulomb Gas system with integer charges and with weight functions for the configurations given by the I_n's.

In a perturbative approach to the quantum field theory defined by the action (3.1), the only ultraviolet divergences which occur in any order in g come from tadpole graphs and can be removed by a normal ordering prescription with respect to an arbitrary mass scale M. All other Feynman graphs are convergent and give rise to finite wave function and mass renormalisation. The coupling constant g does not renormalise.

An essential feature of the Sinh-Gordon theory is its integrability, which in the classical case can be established by means of the inverse scattering method [23]. In order to obtain the expressions of the (classical) conserved currents, let us consider the Euclidean version of the model in terms of the complex coordinates z and $\overline{z}$

$$z \,=\, (x^0 + ix^1) \;;\;\; \overline{z} \,=\, (x^0 - ix^1) \;, \tag{3.6}$$

and define a field $\hat{\phi}(z, \overline{z}, \epsilon)$ which satisfies the following (Bäcklund) equations

$$\begin{aligned} \frac{\partial}{\partial z}(\hat{\phi} + \phi) \,&=\, \frac{m}{g}\epsilon \,\sinh\left(\frac{g}{2}(\hat{\phi} - \phi)\right) \;, \\ \frac{\partial}{\partial \overline{z}}(\hat{\phi} - \phi) \,&=\, \frac{m}{g\epsilon} \,\sinh\left(\frac{g}{2}(\hat{\phi} + \phi)\right) \;. \end{aligned} \tag{3.7}$$

Given that $\phi(z, \overline{z})$ is a solution of the equation of motion originated by (3.1), eqs. (3.7) define a new solution $\hat{\phi}(z, \overline{z}, \epsilon)$ and imply as well the following conservation laws

$$\epsilon^{-1}\,\partial_z \left(\cosh\,\frac{g}{2}(\hat{\phi} + \phi)\right) - \epsilon\,\partial_{\overline{z}} \left(\cosh\,\frac{g}{2}(\hat{\phi} - \phi)\right) = 0 \quad . \tag{3.8}$$

$\hat{\phi}(z, \overline{z}, \epsilon)$ can be expressed in terms of a power series in ϵ

$$\hat{\phi}(z, \overline{z}, \epsilon) \,=\, \sum_{n=0}^{\infty} \phi^{(n)}(z, \overline{z})\,\epsilon^n \tag{3.9}$$

with the fields $\phi^{(n)}(z,\overline{z})$ calculated by using eqs. (3.7). Placing (3.9) into (3.8) and matching equal power in ϵ, one obtains an infinite set of conservation laws

$$\partial_z T_{s+1} \;=\; \partial_{\overline{z}} \Theta_{s-1} \;. \tag{3.10}$$

The corresponding charges $\mathcal{Q}_s$ are given by

$$\mathcal{Q}_s \;=\; \oint \left[T_{s+1}\, dz + \Theta_{s-1}\, d\overline{z} \right] \quad . \tag{3.11}$$

The integer-valued index s which labels the integrals of motion is the spin of the operators. Non-trivial conservation laws are obtained for odd values of s

$$s = 1, 3, 5, 7, \ldots \tag{3.12}$$

In analogy to the Sine-Gordon theory [22], an infinite set of conserved charges $\mathcal{Q}_s$ with spin s given in (3.12) also exists for the quantized version of the Sinh-Gordon theory. They are diagonalised by the asymptotic states with eigenvalues given by

$$\mathcal{Q}_s \,|\beta_1, \ldots, \beta_n> = \; \chi_s \sum_{i=1}^{n} e^{s\beta_i} \,|\beta_1, \ldots, \beta_n> \;\;, \tag{3.13}$$

where χ_s is the normalization constant of the charge $\mathcal{Q}_s$. The existence of these higher integrals of motion precludes the possibility of production processes and hence guarantees that the n-particle scattering amplitudes are purely elastic and factorized into $n(n-1)/2$ two-particle S-matrices. The exact expression for the Sinh-Gordon theory is given by [6]

$$S(\beta, B) \;=\; \frac{\tanh \frac{1}{2}(\beta - i\frac{\pi B}{2})}{\tanh \frac{1}{2}(\beta + i\frac{\pi B}{2})} \;, \tag{3.14}$$

where B is the following function of the coupling constant g

$$B(g) \;=\; \frac{2g^2}{8\pi + g^2} \quad . \tag{3.15}$$

This formula has been checked against perturbation theory in ref. [6] (more recently to higher orders in [8]) and can also be obtained by analytic continuation of the

S-matrix of the first breather of the Sine-Gordon theory [1]. For real values of g the S-matrix has no poles in the physical sheet and hence there are no bound states, whereas two zeros are present at the crossing symmetric positions

$$\beta = \begin{cases} i\frac{\pi B}{2} \\ i\frac{\pi(2-B)}{2} \end{cases} \tag{3.16}$$

The absence of bound states in the Sinh-Gordon model is also supported by the general fusing rule of affine Toda Field Theories [24].

An interesting feature of the S-matrix is its invariance under the map [7]

$$B \to 2 - B \tag{3.17}$$

i.e. under the *strong-weak* coupling constant duality

$$g \to \frac{8\pi}{g}. \tag{3.18}$$

This duality is a property shared by the unperturbed conformal Liouville theory (3.2) [21] and it is quite remarkable that it survives even when the conformal symmetry is broken.

4 Form Factors for the Sinh-Gordon Theory

The Z_2 symmetry of the model is realized by a map σ, whose effect on the elementary field of the theory is $\sigma(\phi) = -\phi$. We assume that it has the same effect on the vertex operator, that is $\sigma(V(\beta)) = -V(\beta)$ together with $\sigma(V(\beta_1)V(\beta_2)) = \sigma(V(\beta_1))\sigma(V(\beta_2))$. According to this symmetry we can label the operators by their Z_2 parity.

For operators which are Z_2-odd the only possible non-zero form factors are those involving an odd number of particles, i.e.

$$F_{2n}^{\mathcal{O}}(\beta_1, \ldots, \beta_{2n}) = 0 \qquad \text{for} \qquad \sigma(\mathcal{O}) = -\mathcal{O}. \tag{4.19}$$

This implies in particular that $\mathcal{O}$ cannot acquire a non-zero vacuum expectation value. On the other hand, for Z_2-even operators the only possible non-zero form factors are those involving an even number of particles, i.e.

$$F^{\mathcal{O}}_{2n+1}(\beta_1,\ldots,\beta_{2n+1}) = 0 \qquad \text{for} \qquad \sigma(\mathcal{O}) = \mathcal{O}. \tag{4.20}$$

The vacuum expectation value of Z_2-even operators can in principle be different from zero.

The simplest representative of the odd sector is given by the (renormalised) field $\phi(x)$ itself. It creates a one-particle state from the vacuum. Our normalization is fixed to be (see sect. 4.3)

$$F^{\phi}_1(\beta) = <0 \mid \phi(0) \mid \beta>_{\text{in}} = \frac{1}{\sqrt{2}} \ . \tag{4.21}$$

For the even sector, an important operator is given by the energy-momentum tensor

$$T_{\mu\nu}(x) = 2\pi \, (: \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}(x) :) \tag{4.22}$$

where $::$ denotes the usual normal ordering prescription with respect an arbitrary mass scale M. Its trace $T^{\mu}_{\mu}(x) = \Theta(x)$ is a spinless operator whose normalization is fixed in terms of its two-particle form factor

$$F^{\Theta}_2(\beta_{12} = i\pi) = {}_{\text{out}}<\beta_1 \mid \Theta(0) \mid \beta_2>_{\text{in}} = 2\pi m^2 \ , \tag{4.23}$$

where m is the physical mass.

In the following we shall compute the form factors of the operators $\phi(x)$ and $\Theta(x)$. This will be sufficient to characterize the basic properties of the model since form factors for other operators can in general be obtained from simple arguments once F^{ϕ}_n and F^{Θ}_n are known. For instance, suppose we want to compute the form factors of the operator $\mathcal{O} =: \sinh g\phi :$. They can be easily computed in terms of the form factor for ϕ. In fact, using eq. (2.14) we have

$$\langle 0|\partial_z\partial_{\bar{z}}\phi(z,\bar{z})|\beta_1\ldots\beta_m\rangle_{\text{in}} = -\frac{m^2}{4}\sum_i e^{\beta_i}\sum_i e^{-\beta_i}\sum_i e^{-ixp_i}F^{\phi}_n(\beta_1,\ldots,\beta_n) \tag{4.24}$$

Employing the equation of motion and choosing $z = \bar{z} = 0$, together with the identities

$$\sum_{i=1}^{n} e^{\beta_i} = \sigma_1^{(n)}(x_1, \ldots, x_n) \qquad , \qquad \sum_{i=1}^{n} e^{-\beta_i} = \frac{\sigma_{n-1}^{(n)}(x_1, \ldots, x_n)}{\sigma_n^{(n)}(x_1, \ldots, x_n)} \tag{4.25}$$

we derive the relation

$$\sigma_n F_n^{\sinh\, g\phi} = g\sigma_1\sigma_{n-1}F_n^{\phi} \ . \tag{4.26}$$

In the last section of this paper we also describe how form factors of operators with higher spin arise from the knowledge of F_n^{Θ} or F_n^{ϕ}.

4.1 Minimal Two-Particle Form Factor

An essential step for the computation the form factors is the determination of $F_{\min}(\beta)$, introduced in (2.20). It satisfies the equations

$$\begin{aligned} F_{\min}(\beta) &= F_{\min}(-\beta)\, S_2(\beta) \ , \\ F_{\min}(i\pi - \beta) &= F_{\min}(i\pi + \beta) \ . \end{aligned} \tag{4.27}$$

As shown in [10], the easiest way to compute $F_{\min}(\beta)$ (up to a normalization $\mathcal{N}$) is to exploit an integral representation of the S-matrix

$$S(\beta) \ = \ \exp\left[\int_0^{\infty} \frac{dx}{x} f(x) \sinh\left(\frac{x\beta}{i\pi}\right)\right] \ . \tag{4.28}$$

Then a solution of (4.27) is given by

$$F_{\min}(\beta) \ = \ \mathcal{N} \ \exp\left[\int_0^{\infty} \frac{dx}{x} f(x) \frac{\sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)}{\sinh x}\right] \ . \tag{4.29}$$

where

$$\hat{\beta} \equiv i\pi - \beta \ . \tag{4.30}$$

For the Sinh-Gordon theory we have

$$F_{\min}(\beta, B) \ = \ \mathcal{N} \ \exp\left[8\int_0^{\infty} \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1-\frac{B}{2})\right) \sinh\frac{x}{2}}{\sinh^2 x} \sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right] \ . \tag{4.31}$$

We choose our normalization to be

$$\mathcal{N} \,=\, \exp\left[-4\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\,\sinh\frac{x}{2}}{\sinh^2 x}\right] \quad . \tag{4.32}$$

The analytic structure of $F_{\min}(\beta,B)$ can be easily read from its infinite product representation in terms of Γ functions

$$F_{\min}(\beta,B) \,=\, \prod_{k=0}^{\infty}\left|\frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+1-\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+1+\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)}\right|^2 \tag{4.33}$$

$F_{\min}(\beta,B)$ has a simple zero at the threshold $\beta = 0$ since $S(0) = -1$ and its asymptotic behaviour is given by

$$\lim_{\beta\to\infty} F_{\min}(\beta,B) = 1. \tag{4.34}$$

It satisfies the functional equation

$$F_{\min}(i\pi+\beta,B)F_{\min}(\beta,B) \,=\, \frac{\sinh\beta}{\sinh\beta+\sinh\frac{i\pi B}{2}} \tag{4.35}$$

which we will use in the next section in order to find a convenient form for the recursive equations of the form factors.

A useful expression for the numerical evaluation of $F_{\min}(\beta,B)$ is given by

$$\begin{aligned} F_{\min}(\beta,B) \,=\, \mathcal{N}\prod_{k=0}^{N-1}&\left[\frac{\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{1}{2}}\right)^2\right)\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{3}{2}-\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat{\beta}/2\pi}{k+1+\frac{B}{4}}\right)^2\right)}{\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{3}{2}}\right)^2\right)\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{1}{2}+\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat{\beta}/2\pi}{k+1-\frac{B}{4}}\right)^2\right)}\right]^{k+1} \\ \times\,\exp&\left[8\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\,\sinh\frac{x}{2}}{\sinh^2 x}(N+1-N\,e^{-2x})\,e^{-2Nx}\,\sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right] \quad . \end{aligned} \tag{4.36}$$

The rate of convergence of the integral may be improved substantially by increasing the value of N. Graphs of $F_{\min}(\beta,B)$ are drawn in fig. 5.

4.2 Parametrization of the n-Particle Form Factors

Since the Sinh-Gordon theory has no bound states, the only poles which appear in any form factor $F_n(\beta_1, \dots, \beta_n)$ are those occurring in every three-body channel. Additional poles in the n-body intermediate channel are excluded by the elasticity of the scattering theory. Using the identity

$$(p_1 + p_2 + p_3)^2 - m^2 \;=\; 8m^2 \cosh\frac{1}{2}\beta_{12} \cosh\frac{1}{2}\beta_{13} \cosh\frac{1}{2}\beta_{23} \;, \tag{4.37}$$

all possible three-particle poles are taken into account by the following parameterization of the function

$$K_n(\beta_1, \dots, \beta_n) \;=\; \frac{Q'_n(\beta_1, \dots, \beta_n)}{\prod\limits_{i<j} \cosh\frac{1}{2}\beta_{ij}} \;, \tag{4.38}$$

where Q'_n is free of any singularity. The second equation in (2.18) implies that Q'_n is $2\pi i$-periodic (anti-periodic) when n is an odd (even) integer. Hence, with a re-definition of Q'_n into Q_n, the general parameterization of the form factor $F_n(\beta_1, \dots \beta_n)$ is chosen to be

$$F_n(\beta_1, \dots, \beta_n) \;=\; H_n\, Q_n(x_1, \dots, x_n) \prod_{i<j} \frac{F_{\min}(\beta_{ij})}{x_i + x_j} \tag{4.39}$$

where $x_i = e^{\beta_i}$ and H_n is a normalization constant. The denominator in (4.39) may be written more concisely as $\det \Sigma$ where the entries of the $(n-1) \times (n-1)$-matrix Σ are given by $\Sigma_{ij} = \sigma^{(n)}_{2i-j}(x_1, \dots, x_n)$.

The functions $Q_n(x_1, \dots, x_n)$ are symmetric polynomials in the variables x_i. As consequence of eq. (2.21), for form factors of spinless operators the total degree should be $n(n-1)/2$ in order to match the total degree of the denominator in (4.39). Form factors of higher spin operators will be considered in sect. 6. The order of the degree of Q_n in each variable x_i is fixed by the nature and by the asymptotic behaviour of the operator $\mathcal{O}$ which is considered.

Employing now the parameterization (4.39), together with the identity (4.35), the recursive equations (2.25) take on the form

$$(-)^n\, Q_{n+2}(-x, x, x_1, \ldots, x_n) \;=\; xD_n(x, x_1, x_2, \ldots, x_n)\, Q_n(x_1, x_2, \ldots, x_n) \quad (4.40)$$

where we have introduced the function

$$D_n = \frac{-i}{4\sin(\pi B/2)} \left(\prod_{i=1}^{n} \left[(x+\omega x_i)(x-\omega^{-1}x_i) \right] - \prod_{i=1}^{n} \left[(x-\omega x_i)(x+\omega^{-1}x_i) \right] \right) \quad (4.41)$$

with $\omega = \exp(i\pi B/2)$. The normalization constants for the form factors of odd and even operators are conveniently chosen to be

$$\begin{aligned} H_{2n+1} &= H_1 \left(\frac{4\sin(\pi B/2)}{F_{\min}(i\pi, B)} \right)^n \\ H_{2n} &= H_2 \left(\frac{4\sin(\pi B/2)}{F_{\min}(i\pi, B)} \right)^{n-1} \end{aligned} \quad (4.42)$$

where H_1 and H_2 are the initial conditions, fixed by the nature of the operator. Using the generating function (2.23) of the symmetric polynomials, the function D_n can be expressed as

$$D_n = \frac{1}{2\sin(\pi B/2)} \sum_{l,k=0}^{n} (-1)^l \sin\left((k-l)\frac{\pi B}{2} \right) x^{2n-l-k} \sigma_l^{(n)} \sigma_k^{(n)}. \quad (4.43)$$

As function of B, D_n is invariant under $B \to -B$. The non-zero terms entering the sum (4.43) are those involving the ratios

$$\frac{\sin(n\,\pi B/2)}{\sin(\pi B/2)}$$

n being an odd number. This means that D_n may only contain powers of $\cos^2(\pi B/2)$.

4.3 LSZ Formula for Form Factors

The aim of this section is to show that the symmetric polynomials Q_{2n+1} entering the form factors of the elementary field $\phi(x)$ can be factorized as

$$Q_{2n+1}(x_1, \ldots, x_{2n+1}) \;=\; \sigma_{2n+1}^{(2n+1)}\, P_{2n+1}(x_1, \ldots, x_{2n+1}) \quad n > 0 \;, \quad (4.44)$$

whereas the analogous polynomials entering the form factors of the trace of the stress-energy tensor can be written as

$$Q_{2n}(x_1,\ldots,x_{2n}) \;=\; \sigma_1^{(2n)}\sigma_{2n-1}^{(2n)}\,P_{2n}(x_1,\ldots,x_{2n}) \quad n>1 \;. \tag{4.45}$$

$P_n(x_1,\ldots,x_n)$ is a symmetric polynomial of total degree $n(n-3)/2$ and of degree $n-3$ in each variable x_i. Using the following property of the elementary symmetric polynomials

$$\sigma_k^{(n+2)}(-x,x,x_1,\ldots,x_n) \;=\; \sigma_k^{(n)}(x_1,x_2,\ldots,x_n) - x^2\sigma_{k-2}^{(n)}(x_1,x_2,\ldots,x_n) \;, \tag{4.46}$$

the recursive equations (4.40) can then be written in terms of the P_n as

$$(-)^{n+1}\,P_{n+2}(-x,x,x_1,\ldots,x_n) \;=\; \frac{1}{x}D_n(x,x_1,x_2,\ldots,x_n)\,P_n(x_1,x_2,\ldots,x_n) \;. \tag{4.47}$$

In order to show the factorization (4.44) for F_{2n+1}^{ϕ}, it is useful to recall the LSZ formula for the form factors of a local operator $\mathcal{O}(x)$

$$F_n(\beta_1,\beta_2,\ldots,\beta_n) \;=\; \left(\frac{1}{\sqrt{2}}\right)^n \lim_{p_i^2\to m^2}\prod_{i=1}^n\left(\frac{p_i^2-m^2}{i}\right) G^{n,\mathcal{O}}(q=-\sum_{i=1}^n p_i,p_1,p_2,\ldots,p_n) \tag{4.48}$$

where

$$\begin{aligned}(2\pi)^2\delta^2(q+\textstyle\sum p_i)G^{n,\mathcal{O}}(q,p_1,p_2,\ldots,p_n) \;=\;& \\ \int\prod_{i=1}^n dx_i\,dy e^{-i\sum p_i x_i}e^{-iqy} &< 0 \mid T(\mathcal{O}(y)\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \mid 0> \quad .\end{aligned} \tag{4.49}$$

The utility of these equations is threefold. First they may allow us to fix the initial condition of the recursive equations (2.25). Second, they permit to study the asymptotic behaviour of the form factors, with a corresponding restriction on the space of solutions. Finally, they provide a tool to check our result through perturbation theory.

When $\mathcal{O}(x)$ is the field $\phi(x)$ itself, the application of eq. (4.48) for $n=1$ gives

$$F_1^{\phi}(\beta) \;=< 0 \mid \phi(0) \mid \beta >_{\rm in} = \;\frac{1}{\sqrt{2}}\lim_{p^2\to m^2}\frac{p^2-m^2}{i}G^2(p) \tag{4.50}$$

which provides the initial condition

$$F_1^{\phi}(\beta) \,=\, \frac{1}{\sqrt{2}} \,. \tag{4.51}$$

It is now easy to establish that the form factors F_{2n+1}^{ϕ} of the elementary field $\phi(x)$ are proportional to $\sigma_{2n+1}^{(2n+1)}$. The reason is that from any Feynman diagram which enters F_{2n+1}^{ϕ} we can factorize the propagator

$$\frac{i}{q^2-m^2}\,\Big|_{q=-\sum p_i,\, p_i^2=m^2} \tag{4.52}$$

that, written in terms of the variables x_i, becomes proportional to $\sigma_{2n+1}^{(2n+1)}$

$$\frac{i}{q^2-m^2}\,|_{q=-\sum p_i,\, p_i^2=m^2} = \,\frac{i}{m^2}\frac{\sigma_{2n+1}^{(2n+1)}}{\sigma_1^{(2n+1)}\sigma_{2n}^{(2n+1)}-\sigma_{2n+1}^{(2n+1)}} \,. \tag{4.53}$$

The presence of the propagator (4.52) in front of any form factor of the elementary field $\phi(x)$ also implies that F_{2n+1}^{ϕ} behaves asymptotically as

$$F_{2n+1}(\beta_1,\beta_2,\ldots,\beta_{2n+1}) \to 0 \;\; \text{as}\; \beta_i \to +\infty \;\; \beta_{j\neq i} \;\text{fixed.} \tag{4.54}$$

In fact, the propagator (4.52) goes to zero in this limit whereas the remaining expression of the Feynman graphs entering F_{2n+1} is a perturbative series which starts from the tree level vertex diagram shown in fig. 6, which is a constant. Other tree level contributions at the lowest order and higher order corrections are either finite or they vanish in the limit (4.54). In fact, by dimensional analysis they must have external momenta in the denominator in order to compensate the increasing power of the mass in the coupling constants.

In order to prove the factorization (4.45) for the form factors F_{2n}^{Θ} of the trace of the stress-energy tensor, let us consider the conservation laws satisfied by this operator

$$\partial_{\bar z} T(z,\bar z) + \partial_z \Theta(z,\bar z) = 0 \quad , \qquad\qquad \partial_z \bar T(z,\bar z) + \partial_{\bar z} \Theta(z,\bar z) = 0 \tag{4.55}$$

where T ($\bar T$) is the component of the stress-energy tensor which in the conformal limit becomes holomorphic (anti-holomorphic). Using eq. (2.14), the identities (4.25

together with (4.55), we obtain

$$\sigma_1^{(2n)}\sigma_{2n}^{(2n)}F_{2n}^{T}(\beta_1,\ldots,\beta_{2n}) = \sigma_{2n-1}^{(2n)}F_{2n}^{\Theta}(\beta_1,\ldots,\beta_{2n}) \quad (4.56)$$

$$\sigma_{2n-1}^{(2n)}F_{2n}^{\bar{T}}(\beta_1,\ldots,\beta_{2n}) = \sigma_1^{(2n)}\sigma_{2n}^{(2n)}F_{2n}^{\Theta}(\beta_1,\ldots,\beta_{2n}). \quad (4.57)$$

Since F_{2n}^{T}, $F_{2n}^{\bar{T}}$ and F_{2n}^{Θ} are expected to have the same analytic structure, we conclude that $F_{2n}^{\Theta}(\beta_1,\ldots,\beta_{2n})$ is proportional to the product $\sigma_1^{(2n)}\sigma_{2n-1}^{(2n)}$ for $n>2$.

4.4 Solutions of the Recursive Equations

Let us summarize the analysis carried out in the previous sections. The form factors F_{2n+1}^{ϕ} $(n>0)$ of the elementary field $\phi(x)$ are given by

$$F_{2n+1}^{\phi}(\beta_1,\ldots,\beta_{2n+1}) = \frac{1}{\sqrt{2}}\left(\frac{4\sin(\pi B/2)}{F_{\min}(i\pi,B)}\right)^{n}\sigma_{2n+1}^{(2n+1)}P_{2n+1}(x_1,\ldots,x_{2n+1})\prod_{i<j}\frac{F_{\min}(\beta_{ij})}{x_i+x_j} \quad (4.58)$$

and the normalization of the field is fixed by

$$F_1^{\phi} = \frac{1}{\sqrt{2}}\ . \quad (4.59)$$

The form factors F_{2n}^{Θ} $(n>1)$ of the trace of the stress-energy tensor $\Theta(x)$ are given by

$$F_{2n}^{\Theta}(\beta_1,\ldots,\beta_{2n}) = \frac{2\pi m^2}{F_{\min}(i\pi)}\left(\frac{4\sin(\pi B/2)}{F_{\min}(i\pi)}\right)^{n-1}\sigma_1^{(2n)}\sigma_{2n-1}^{(2n)}P_{2n}(x_1,\ldots,x_{2n})\prod_{i<j}\frac{F_{\min}(\beta_{ij})}{x_i+x_j} \quad (4.60)$$

where the normalization is fixed by the matrix element of $\Theta(0)$ between the two-particle state and the vacuum

$$F_2^{\Theta}(\beta_{12}) = 2\pi m^2\frac{F_{\min}(\beta_{12})}{F_{\min}(i\pi)}\ . \quad (4.61)$$

Notice that (4.60) for $n=0$ leads to the expectation value of Θ on the vacuum

$$<0|\Theta(0)|0> = \frac{\pi m^2}{2\sin(\pi B/2)}\ . \quad (4.62)$$

Using the recursive equations (4.47) and the transformation property of the elementary symmetric polynomials (4.46), the explicit expressions of the first polynomials $P_n(x_1, \ldots, x_n)$ are given by[‡]

$$\begin{aligned}
P_3(x_1, \ldots, x_3) &= 1 \\
P_4(x_1, \ldots, x_4) &= \sigma_2 \\
P_5(x_1, \ldots, x_5) &= \sigma_2\sigma_3 - c_1^2\sigma_5 \\
P_6(x_1, \ldots, x_6) &= \sigma_3(\sigma_2\sigma_4 - \sigma_6) - c_1^2(\sigma_4\sigma_5 + \sigma_1\sigma_2\sigma_6) \\
P_7(x_1, \ldots, x_7) &= \sigma_2\sigma_3\sigma_4\sigma_5 - c_1^2(\sigma_4\sigma_5^2 + \sigma_1\sigma_2\sigma_5\sigma_6 + \sigma_2^2\sigma_3 - c_1^2\sigma_2\sigma_5) + \\
&\quad -c_2(\sigma_1\sigma_6\sigma_7 + \sigma_1\sigma_2\sigma_4\sigma_7 + \sigma_3\sigma_5\sigma_6) + c_1c_2^2\sigma_7^2
\end{aligned} \tag{4.63}$$

where $c_1 = 2\cos(\pi B/2)$ and $c_2 = 1 - c_1^2$. Expression of the higher P_n are easily computed by an iterative use of eqs. (4.40). For practical application the first representatives of P_n are sufficient to compute with a high degree of accuracy the correlation functions of the fields. In fact, the n-particle term appearing in the correlation function of the fields (2.28) behaves as $e^{-n(mr)}$ and for quite large values of mr the correlator is dominated by the lowest number of particle terms. This conclusion is also confirmed by an application of the c-theorem which is discussed in sect. 6. Nevertheless, it is interesting to notice that closed expressions for P_n can be found for particular values of the coupling constant, as we demonstrate in the next subsections.

4.4.1 The Self-Dual Point

The self-dual point in the coupling constant manifold has the special value

$$B\left(\sqrt{8\pi}\right) = 1 \ . \tag{4.64}$$

[‡]The upper index of the elementary symmetric polynomials entering P_n is equal to n and we suppress it, in order to simplify the notation.

The two zeros of the S-matrix merge together and the function $D_n(x, x_1, x_2, \ldots, x_n)$ acquires the particularly simple form

$$D_n(x|x_1, x_2 \ldots, x_n) \;=\; \left(\sum_{k=0}^{n}(-1)^{k+1}\sin\frac{k\pi}{2}x^{n-k}\sigma_k^{(n)}\right)\left(\sum_{l=0}^{n}(-1)^{l}\cos\frac{l\pi}{2}x^{n-l}\sigma_l^{(n)}\right) \quad . \tag{4.65}$$

In this case the general solution of the recursive equations (4.47) is given by

$$P_n(x_1, x_2, \ldots, x_n) \;=\; \det\mathcal{A}(x_1, x_2, \ldots, x_n) \tag{4.66}$$

where $\mathcal{A}$ is an $(n-3)\times(n-3)$ matrix whose entries are

$$\mathcal{A}_{ij}(x_1, x_2, \ldots, x_n) \;=\; \sigma_{2j-i+1}^{(n)}\,\cos^2\left[(i-j)\frac{\pi}{2}\right] \quad , \tag{4.67}$$

i.e.

$$\mathcal{A} = \begin{pmatrix} \sigma_2 & 0 & \sigma_6 & 0 & \cdots \\ 0 & \sigma_3 & 0 & \sigma_7 & \cdots \\ 1 & 0 & \sigma_4 & 0 & \cdots \\ 0 & \sigma_1 & 0 & \sigma_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.68}$$

This can be proved by exploiting the properties of determinants. i.e. their invariance under linear combinations of the rows and the columns. Let us consider the $(n-1)\times(n-1)$ matrix associated to $P_{n+2}(-x, x, x_1, \ldots, x_n)$

$$\mathcal{A}_{ij} \;=\; \left(\sigma_{2j-i+1}^{(n)} - x^2\,\sigma_{2j-i-1}^{(n)}\right)\cos^2\left[(i-j)\frac{\pi}{2}\right] \quad , \tag{4.69}$$

where eq. (4.46) was used. Adding successively x^2 times the row $(i+2)$ to row i (starting with $i=1$), we obtain for the entries of the matrix $\mathcal{A}$

$$\mathcal{A}_{ij} \;=\; \left(\sigma_{2j-i+1}^{(n)} - x^4\,\sigma_{2j-i-3}^{(n)}\right)\cos^2\left[(i-j)\frac{\pi}{2}\right]. \tag{4.70}$$

Adding now x^4 times of the i^{th} column to column $(i+2)$ (starting with $i=1$), we obtain the following matrix:

$$\mathcal{A}^{(n-1)\times(n-1)} = \begin{pmatrix} & 0 & 0 \\ \mathcal{A}^{(n-3)\times(n-3)} & \vdots & \vdots \\ & 0 & \vdots \\ * \cdots * & \mathcal{A}_{(n-2)(n-2)} & 0 \\ * \cdots * & 0 & \mathcal{A}_{(n-1)(n-1)} \end{pmatrix} \tag{4.71}$$

where the entries in the lower right corner are given by

$$\begin{aligned} \mathcal{A}_{(n-2)(n-2)} &= \sum_{k=0}^{n} (-1)^k \cos\frac{k\pi}{2} x^{n-k-1} \sigma_k^{(n)} \\ \mathcal{A}_{(n-1)(n-1)} &= \sum_{l=0}^{n} (-1)^{l+1} \sin\frac{l\pi}{2} x^{n-l} \sigma_l^{(n)} \quad . \end{aligned}$$

Developing the determinant of this matrix with respect to the last two columns and taking into account eqs. (4.65) and (4.66), we obtain the right hand side of equation (4.47), Q.E.D.

4.4.2 The "Inverse Yang-Lee" Point

A closed solution of the recursive equations (4.47) is also obtained for

$$B\left(2\sqrt{\pi}\right) = \frac{2}{3} \ . \tag{4.72}$$

The reason is that, for this particular value of the coupling constant the S-matrix of the Sinh-Gordon theory coincides with the inverse of the S-matrix $S_{\rm YL}(\beta)$ of the Yang-Lee model [4] or, equivalently

$$S(\beta, -\frac{2}{3}) \ = \ S_{\rm YL}(\beta) \ . \tag{4.73}$$

Since the recursive equations (4.47) are invariant under $B \to -B$ (see sect. 4.2), a solution is provided by the same combination of symmetric polynomials found for the Yang-Lee model [13, 15], i.e.

$$P_n(x_1, x_2, \ldots, x_n) \ = \ \det \mathcal{B}(x_1, x_2, \ldots, x_n) \tag{4.74}$$

with the following entries of the $(n-3)\times(n-3)$-matrix $\mathcal{B}$

$$\mathcal{B}_{ij} \;=\; \sigma_{3j-2i+1} \;\; . \tag{4.75}$$

The proof is similar to the one of the previous section and exploits the invariance of a determinant under linear combinations of the rows and the columns. In this case the function D_n is most conveniently expressed as determinant of a 2×2-matrix

$$D_n = \det \begin{pmatrix} \left(\sum\limits_{l=0}^{n}(-1)^l \cos\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) & \left(\sum\limits_{l=0}^{n}\cos\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) \\ \left(\sum\limits_{l=0}^{n}(-1)^l \sin\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) & \left(\sum\limits_{l=0}^{n}\sin\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) \end{pmatrix} \tag{4.76}$$

Let us consider the $(n-1)\times(n-1)$-matrix entering the expression $P_{n+2}(-x,x,x_1,\ldots,x_n)$, i.e.

$$\mathcal{B}_{ij} \;=\; \sigma_{3j-2i+1} - x^2\sigma_{3j-2i-1}, \tag{4.77}$$

By adding successively the i^{th} row to row $(i-1)$ (starting with $i=(n-1)$), we obtain

$$\mathcal{B}_{ij} \;=\; \sigma_{3j-2i+1} - x^{2(n-i)}\sigma_{3j-2n+1}.$$

Then by adding successively x^6 times the i^{th} column to column $(i+2)$, starting with $i=1$, the entries for the matrix $\mathcal{B}$ read

$$\mathcal{B}_{ij} \;=\; \sum_{l=0}\sigma_{3j-2i-6l+1}x^{6l} - x^{2(n-i+3l)}\sigma_{3j-2n-6l+1}. \tag{4.78}$$

Subtracting x^6 times of the row $(i+3)$ from row i (starting with $i=1$) we finally obtain the matrix

$$\mathcal{B}^{(n-1)\times(n-1)} = \begin{pmatrix} & 0 & 0 \\ \mathcal{B}^{(n-3)\times(n-3)} & \vdots & \vdots \\ & 0 & 0 \\ * \cdots * & \mathcal{B}_{(n-2)(n-2)} & \mathcal{B}_{(n-2)(n-1)} \\ * \cdots * & \mathcal{B}_{(n-1)(n-2)} & \mathcal{B}_{(n-1)(n-1)} \end{pmatrix} \tag{4.79}$$

where the entries of the (2×2) matrix in the lower right corner are still given by (4.78). It is easy to prove that the determinant of this (2×2) matrix in the lower

right corner is equal to (4.76). Therefore, with the definition (4.74), the determinant of $\mathcal{B}^{(n-1)\times(n-1)}$ gives rise the right hand side of (4.47). Q.E.D.

5 Form Factors for Descendent Operators

In this section we investigate the effect of the Lorentz transformation on the space of solutions for all form factors denoted by $\mathcal{P}$. This problem has been firstly addressed by Cardy and Mussardo [16] for the space of descendant operators of the Ising model.

The space of the form factors $\mathcal{P}$ can be decomposed as

$$\mathcal{P} = \bigoplus_s \ \mathcal{P}_s \quad , \tag{5.80}$$

meaning that $F_n^{\mathcal{O}} \in \mathcal{P}_s$ if $\mathcal{O}$ has spin s. On the rapidity variables a Lorentz transformation is realized by $\beta_i \to \beta_i + \varepsilon$, i.e. $x_i \to e^{\varepsilon} x_i$. Since the elementary symmetric polynomials are homogeneous functions of x_i, under a Lorentz transformation they transform as

$$\sigma_n(x_1, ..., x_n) \to e^{n\varepsilon} \sigma_n(x_1, ..., x_n). \tag{5.81}$$

Hence, given a form factor $F_n^{\mathcal{O}}$ of an operator $\mathcal{O}$ with spin s which satisfies Watson's equations, a new function in $\mathcal{P}_{s+s'}$ which still satisfies Watson's equations can be defined by

$$F_n^{\mathcal{O}'}(x_1, \ldots, x_n) = I_n^{s'}(x_1, \ldots, x_n) F_n^{\mathcal{O}}(x_1, \ldots, x_n), \tag{5.82}$$

provided that I_n^s is composed out of elementary symmetric polynomials. Additional constraints on the I_n^s are imposed by their invariance under the kinematic residue equation

$$I_{n+2}^s(-x, x, x_1, \ldots, x_n) = I_n^s(x_1, \ldots, x_n) \ . \tag{5.83}$$

A basis in the space of solutions of eq. (5.83) is given by symmetric polynomials I_n^s satisfying the recursion relations [16]

$$\sigma_{2k+1}^{(n)} \ = \ I_n^{2k+1} + \sigma_2^{(n)} \, I_n^{2k-1} + \sigma_4^{(n)} \, I_n^{2k-3} + \ldots + \sigma_{2k}^{(n)} \, I_n^1 \ , \tag{5.84}$$

where s is equal to the spin of the conserved charges (3.12). A closed expression of I_n^s as been obtained in [17]

$$I_n^{2s-1} = (-1)^{s+1} \det \mathcal{I} \tag{5.85}$$

where the entries of the $(s \times s)$-matrix $\mathcal{I}$ for $j = 1, \ldots, s$ and $i = 2, \ldots, s$ are

$$\mathcal{I}_{1j} = \sigma_{2j-1} \qquad\qquad \mathcal{I}_{ij} = \sigma_{2j-2i+2} \tag{5.86}$$

i.e.

$$\mathcal{I} = \begin{pmatrix} \sigma_1 & \sigma_3 & \sigma_5 & \sigma_7 & \ldots & \sigma_{2s-1} \\ 1 & \sigma_2 & \sigma_4 & \sigma_6 & \ldots & \sigma_{2s-2} \\ 0 & 1 & \sigma_2 & \sigma_4 & \ldots & \sigma_{2s-4} \\ 0 & 0 & 1 & \sigma_2 & \ldots & \sigma_{2s-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \tag{5.87}$$

The determinant of $\mathcal{I}$ will always be of order $2s-1$ as required.

As it was first noticed in [4], eqs. (5.82) naturally provides a grading in the space of matrix elements of local operators in an integrable massive field theory. In fact, given an invariant polynomial I_n^s, eq. (5.82) defines form factors of an operator $\mathcal{O}'_s$ which, borrowing the terminology of Conformal Field Theories [26], is natural to call *descendant operator* of the spinless field $\mathcal{O}$. In particular, choosing $\mathcal{O}$ to be the trace of the stress-energy tensor, the form factors defined by eq. (5.82) are related to the matrix elements of the higher conserved currents, as can be easily seen by eq. (3.13) and by the fact that the symmetric polynomials which appear as eigenvalues of the conserved charges $\mathcal{Q}_s$

$$s_k \,=\, x_1^k + x_2^k + \ldots + x_n^k \;, \tag{5.88}$$

can be expressed in terms of the invariant polynomials I_n^s. Indeed they satisfy the recursive relation

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \ldots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k \,=\, 0 \;, \tag{5.89}$$

that, together with eq. (5.84), permits to express s_k in terms of the invariant polynomials I_n^s.

6 Form factors and c-theorem

As mentioned in sect. 3, the Sinh-Gordon model can be regarded as deformation of the free massless theory with central charge $c = 1$. This fixed point governs the ultraviolet behaviour of the model whereas the infrared behaviour corresponds to a massive field theory with central charge $c = 0$. Going from the short- to large-distances, the variation of the central charge is dictated by the c-theorem of Zamolodchikov [27]. An integral version of this theorem has been derived by Cardy [28] and related to the spectral representation of the two-point function of the trace of the stress-energy tensor in [29, 30], i.e.

$$\Delta c \;=\; \int_0^\infty d\mu \, c_1(\mu) \;, \tag{6.1}$$

where $c_1(\mu)$ is given by

$$\begin{aligned} c_1(\mu) \;&=\; \frac{6}{\pi^2}\frac{1}{\mu^3}\mathrm{Im}\, G(p^2 = -\mu^2) \;, \\ G(p^2) \;&=\; \int d^2x \, e^{-ipx} \; < 0|\Theta(x)\Theta(0)|0 >_{\mathrm{conn}} \quad . \end{aligned} \tag{6.2}$$

Inserting a complete set of in-state into (6.2), we can express the function $c_1(\mu)$ in terms of the form factors F_{2n}^Θ

$$\begin{aligned} c_1(\mu) \;&=\; \frac{12}{\mu^3}\sum_{n=1}^\infty \frac{1}{(2n)!}\int \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n}} \; \mid F_{2n}^\Theta(\beta_1, \ldots, \beta_{2n}) \mid^2 \\ &\quad \times \delta(\sum_i m \sinh\beta_i)\, \delta(\sum_i m\cosh\beta_i - \mu) \;\; . \end{aligned} \tag{6.3}$$

For the Sinh-Gordon theory $\Delta c = 1$ and it is interesting to study the convergence of this series increasing the number of intermediate particles. For the two-particle contribution, we have the following expression

$$\Delta c^{(2)} \;=\; \frac{3}{2F_{\min}^2(i\pi)} \int_0^\infty \frac{d\beta}{\cosh^4\beta} \, |F_{\min}(2\beta)|^2 \;\; . \tag{6.4}$$

The numerical results for different values of the coupling constant $g^2/4\pi$ are listed in Table 1. It is evident that the sum rule is saturated by the two-particle form factor also for large values of the coupling constant. Hence, the expansion in the number of intermediate particles results in a fast convergent series, as it is confirmed by the computation of the next terms involving the form factor with four and six particles.

7 Conclusions

The computation of the Green functions is a central problem in a Quantum Field Theory. For integrable models, a promising approach to this question is given by the bootstrap principle applied to the computation of the matrix elements of local operators. In this paper we have investigated the form factors of the most representative fields of the Z_2 sectors of the Sinh-Gordon model, i.e. the field $\phi(x)$ and the trace of the stress-energy tensor $\Theta(x)$. The simplicity of the Sinh-Gordon model permits clarification of the basic properties of the local operators and their matrix elements in a QFT, without being masked by algebraic complexities due to the structure of the bound states. Compared to the usual method of computing correlation functions in terms of a perturbative series in the coupling constant, the form factor approach is extremely advantageous for two reasons. Firstly, the coupling constant dependence of the correlation functions is encoded (to all orders in g) into the expression of $F_{\min}(b, B)$, eq. (4.31), and into the solutions of the recursive equations (2.25) for the pre-factors K_n entering the form factors, eq. (2.20). Secondly, even for not large values of the distances, the resulting expressions of the correlation functions as an infinite series over the multi-particle form factors are actually dominated by the lowest number of particle terms and therefore present a very fast rate of convergence.

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Figure Captions

Figure 1 . Form factors of the operator $\mathcal{O}(0)$.

Figure 2 . Kinematical recursive equation for the form factor $\mathcal{F}_n$

Figure 3 . Bound state pole in scattering amplitude.

Figure 4 . Bound state recursive equation for the form factor $\mathcal{F}_n$.

Figure 5 . Graphs of $|F_{\min}(\beta, B)/\mathcal{N}|^2$ as function of β for different values of $B(g)$.

Figure 6 . Lowest terms in the perturbative expression of the form factors of the elementary field $\phi(0)$. $G^{(2n+1)}$ is the Green function with $2n+1$ external legs.

Table Caption

Table 1 . The first two-particle term entering the sum rule of the c-theorem.

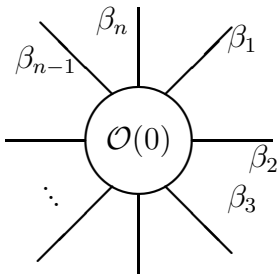

Figure 1

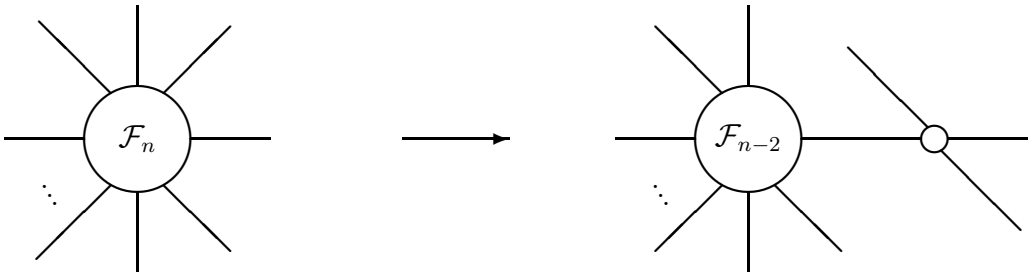

Figure 2

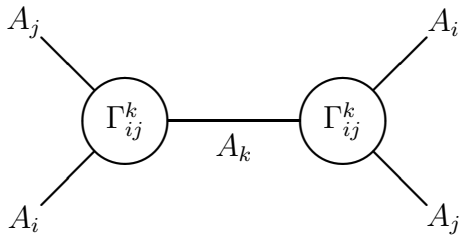

Figure 3

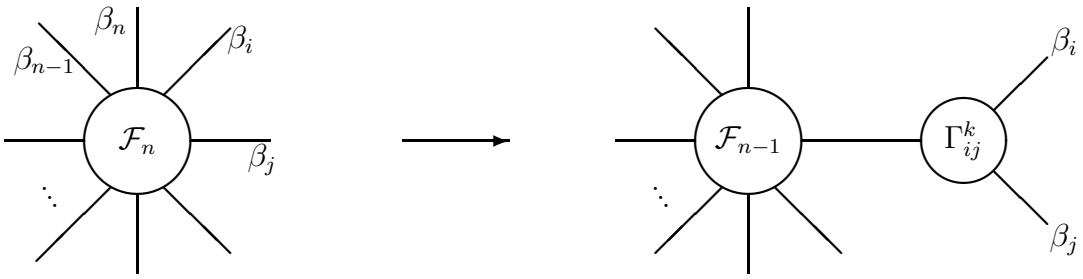
β_n
β_i
β_{n-1}
$\mathcal{F}_n$
β_j
$\mathcal{F}_{n-1}$
Γ_{ij}^k
β_i
β_j

Figure 4

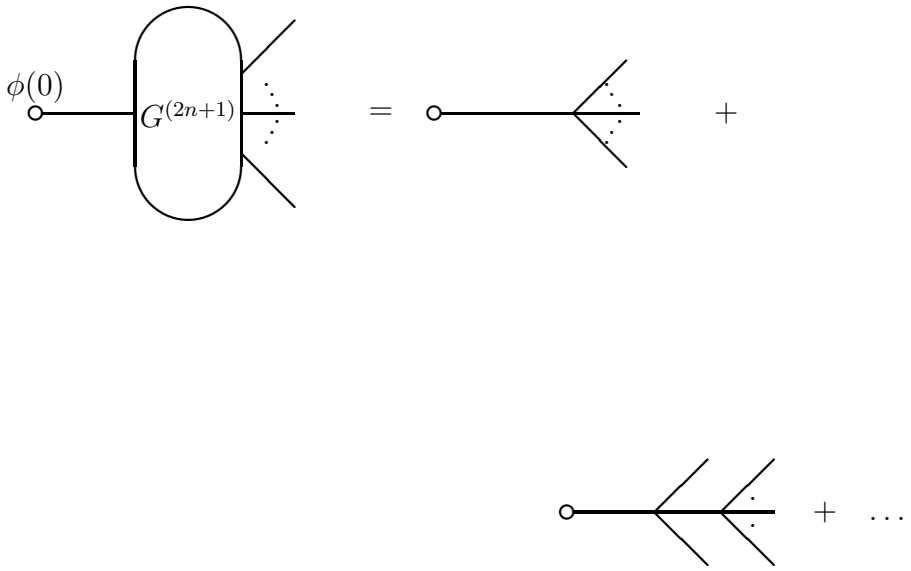

Figure 6

B	$\frac{g^2}{4\pi}$	$\Delta\, c^{(2)}$
$\frac{1}{500}$	$\frac{2}{999}$	0.9999995
$\frac{1}{100}$	$\frac{2}{199}$	0.9999878
$\frac{1}{10}$	$\frac{2}{19}$	0.9989538
$\frac{3}{10}$	$\frac{6}{17}$	0.9931954
$\frac{2}{5}$	$\frac{1}{2}$	0.9897087
$\frac{1}{2}$	$\frac{2}{3}$	0.9863354
$\frac{2}{3}$	1	0.9815944
$\frac{7}{10}$	$\frac{14}{13}$	0.9808312
$\frac{4}{5}$	$\frac{4}{3}$	0.9789824
1	2	0.9774634

Table 1