Quasi-hereditary quotients of finite Chevalley groups and Frobenius kernels

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1 Introduction and notations

1.1. Let G be a semisimple connected simply connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$. Assume G is defined and split over the prime subfield $\mathbf{F_p}$. For each integer $n \geq 1$, denote by G_n the n-th Frobenius kernel of G. The representation theory of G_n is equivalent to the representation theory of the finite dimensional algebra $\mathcal{U}_n$ given by the dual of its coordinate algebra. For $q = p^n$ we also have the finite subgroups $G(q)$ of G consisting of $\mathbf{F_q}$-rational points of G. We are interested in relating the representation theory of $\mathcal{U}_n$ and $G(q)$ over k with the quasi-hereditary algebras arising from the rational representations of G.

The category of G-modules which are bounded in a certain sense is equivalent to the module category of a finite dimensional quasi-hereditary algebra. These algebras are very well understood, see for example Ringel [12] in the general context of finite dimensional algebras and Donkin [7] [6] in this context.

The group algebra $kG(q)$ and the algebra $\mathcal{U}_n$ are not quasi-hereditary unless they are semisimple, as they are self-injective algebras. But in this paper we find quotients of $kG(q)$ and $\mathcal{U}_n$ which are quasi-hereditary (see section 2 for a precise statement). We give two different proofs of our result. The first one, given in section 3, only works when the prime p is large enough. It uses the fact that the indecomposable projective $\mathcal{U}_n$-modules, resp. $kG(q)$-modules, can be obtained by restricting certain tilting modules for the group G. The second proof, given in section 4, is due to Stephen Donkin. It is a direct proof using coalgebras and it works without any restriction on the prime p. We are extremely grateful to Stephen Donkin for allowing us to include his proof in this paper.

1.2. Notation Let $R = k[G]$ be the coordinate algebra of the group G. It has a structure of Hopf algebra. Consider the following Hopf ideals of R:

$$I = \{f \in R \,|\, f(1) = 0\},$$
$$I^{[p^n]} = \{\sum_{f \in I} Rf^n\}.$$

The coordinate algebra of the n-th Frobenius kernel is given by $k[G_n] := R/I^{[p^n]}$ and the algebra $\mathcal{U}_n := k[G_n]^*$. It is a finite dimensional self-injective algebra (see [8]I.8.10).

Fix a maximal split torus T in G and let $X(T)$ denote its character lattice. Let Φ be the root system of G with respect to T. Fix Φ^- (resp. Φ^+) the set of positive (resp. negative) roots and denote by Δ the set of simple roots. Let B be the Borel subgroup corresponding to the negative roots. This partition of Φ defines a partial ordering on $X(T)$ as follows. For $\lambda, \mu \in X(T)$, we say that $\lambda \geq \mu$ if $\lambda - \mu$ can be written as a sum of simple roots with non-negative integer coefficients. Let $W := N_G(T)/T$ be the Weyl group. We denote the longest element in W by w_0. The Weyl group W acts on $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Fix an inner product $\langle ., . \rangle$ on $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$ invariant under the action of W. For each root $\alpha \in \Phi$, denote by $\alpha^v = 2\alpha / \langle \alpha, \alpha \rangle$ the corresponding coroot. Define ρ to be half the sum of all positive roots in Φ. The Coxeter number of Φ is given by

$$h := \max\{\langle \rho, \beta^v \rangle + 1 \,|\, \beta \in \Phi^+\}.$$

It is the maximum of the Coxeter numbers of the connected components of Φ. If Φ is connected, we denote by α_0 the highest short root of Φ.
The set of dominant weights is defined by

$$X^+(T) = \{\lambda \in X(T) \,|\, \langle \lambda, \alpha^v \rangle \geq 0 \ \forall \alpha \in \Delta\}.$$

The simple G-modules are indexed by the set of dominant weights $X^+(T)$ and denoted by $L(\lambda)$, $\lambda \in X^+(T)$. They are given by $L(\lambda) = \mathrm{soc}\nabla(\lambda)$ where $\nabla(\lambda)$ is the induced module $\mathrm{Ind}_B^G \lambda$. The Weyl modules $\Delta(\lambda)$ are defined to be the contravariant duals of the induced modules

$$\Delta(\lambda) := \nabla(-w_0\lambda)^*.$$

When $\lambda = (p^n - 1)\rho$, we have $\nabla((p^n - 1)\rho) = \Delta((p^n - 1)\rho) = L((p^n - 1)\rho)$, this module is called the n-th Steinberg module and is denoted by St_n.
We say that a rational G-module M has a ∇-filtration if M has a filtration

$$0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_k = M$$

such that each quotient $M_i/M_{i-1} = \nabla(\mu)$ for some $\mu \in X^+(T)$. We define Δ-filtration similarly. The rational G-modules having both a ∇-filtration and a Δ-filtration are called tilting modules. It can be shown that the indecomposable tilting modules are indexed by dominant weights, we denote them by $T(\lambda)$, $\lambda \in X^+(T)$.

The set of p^n-restricted weights is given by

$$X_n = \{\lambda \in X^+(T) \mid \langle \lambda, \alpha^v \rangle < p^n \ \forall \alpha \in \Delta\}.$$

A complete set of non-isomorphic $kG(q)$-modules, resp. $\mathcal{U}_n$-modules, is obtained by restricting the set of simple G-modules corresponding to p^n-restricted weights

$$\{L(\lambda) \mid \lambda \in X_n\}.$$

We denote by $U(\lambda)$, resp. $Q(\lambda)$, the projective cover (injective hull), of $L(\lambda)$ in the category of $kG(q)$-modules, resp. $\mathcal{U}_n$-modules.

1.3 Truncation functors O^π and O_π and the algebras $S(\pi)$

In this section we start by describing the functors O_π and O^π. We then introduce the algebras $S(\pi)$ defined by Donkin (see [7]). Let π be a finite subset of the set of dominant weights $X^+(T)$. We say that a rational G-module V belongs to π if all its composition factors $L(\mu)$ satisfy $\mu \in \pi$. For a rational G-module M, we define $O_\pi(M)$ to be the largest submodule of M belonging to π. Similarly, we define $O^\pi(M)$ to be the smallest submodule of M such that the quotient module belongs to π. The coordinate algebra $k[G]$ has a structure of rational G-module so we can form $A(\pi) := O_\pi(k[G])$. This is a subcoalgebra of $k[G]$. We define the finite dimensional algebra $S(\pi) = A(\pi)^*$. There is an equivalence between the category of left rational G-modules belonging to π, the category of right $A(\pi)$-comodules and the category of left $S(\pi)$-modules. So a complete set of non-isomorphic simple $S(\pi)$-modules is given by

$$\{L(\lambda) \mid \lambda \in \pi\}.$$

For each $\lambda \in \pi$, we denote by $P_\pi(\lambda)$ the projective cover of $L(\lambda)$ as an $S(\pi)$-module. Now suppose that we have a subset $\pi' \subset \pi$ then it is easy to check that for $\lambda \in \pi'$ the projective cover $P_{\pi'}(\lambda)$ of $L(\lambda)$ as an $S(\pi')$-module is given by

$$P_{\pi'}(\lambda) = P_\pi(\lambda)/O^{\pi'}(P_\pi(\lambda)).$$

We say that a finite subset π of $X^+(T)$ is saturated in $X^+(T)$ if whenever $\lambda \in \pi$ and $\mu \in X^+(T)$ with $\mu \leq \lambda$ we have $\mu \in \pi$. In this case, it can be shown (see [7](2.2h)) that the finite dimensional algebra $S(\pi)$ is quasi-hereditary in the sense of Cline Parshall and Scott (see [4]). In particular, for $\lambda \in \pi$ the standard modules are given by the Weyl modules $\Delta(\lambda)$, the costandard modules are the induced modules $\nabla(\lambda)$ and the tilting modules are given by the $T(\lambda)$'s (see [6]).

2 Results

Theorem 2.1 *Let $\pi \subseteq X_n$ then there is an ideal J of the algebra $\mathcal{U}_n$ such that the quotient $\mathcal{U}_n/J$ is Morita equivalent to $S(\pi)$. In particular, if π is a saturated subset of $X^+(T)$ then we obtain a quasi-hereditary quotient of $\mathcal{U}_n$.*

Theorem 2.2 *Let $\pi \subseteq X_n$ then there is an ideal I of the group algebra $kG(q)$ such that the quotient $kG(q)/I$ is Morita equivalent to $S(\pi)$. In particular, if π is a saturated subset of $X^+(T)$ then we obtain a quasi-hereditary quotient of $kG(q)$.*

We obtain immediately the following corollary.

Corollary 2.1 *There is an ideal J of $\mathcal{U}_n$ and an ideal I of $kG(q)$ such that we have the following Morita equivalence*

$$\mathcal{U}_n/J \sim_{\mathcal{M}} kG(q)/I \sim_{\mathcal{M}} S(X_n).$$

3 First proof (when $p \geq 2h-2$)

This proof only works for $p \geq 2h-2$.
We will use the following general fact about Morita equivalence (see for example [2](2.2)): Let A be a finite dimensional algebra and let $\{P_1, ..., P_l\}$ be

a complete set of indecomposable projective A-modules, then

$$\mathrm{End}_A\left(\bigoplus_{i=1}^{l} P_i\right)^{op} \sim_{\mathcal{M}} A.$$

So in order to prove Theorems 2.1 and 2.2, we shall construct surjective algebra homomorphisms:

$$\Phi_1 : \mathrm{End}_{\mathcal{U}_n}\left(\bigoplus_{\lambda\in X_n} Q(\lambda)\right) \longrightarrow \quad \mathrm{End}_{S(\pi)}\left(\bigoplus_{\lambda\in\pi} P_\pi(\lambda)\right),$$

$$\Phi_2 : \mathrm{End}_{kG(q)}\left(\bigoplus_{\lambda\in X_n} U(\lambda)\right) \longrightarrow \quad \mathrm{End}_{S(\pi)}\left(\bigoplus_{\lambda\in\pi} P_\pi(\lambda)\right).$$

The next two theorems tell us how to obtain the projective $\mathcal{U}_n$-modules and $kG(q)$-modules by restricting certain G-modules. For $\lambda \in X_n$, denote by $\bar{\lambda} := 2(p^n - 1)\rho + w_0\lambda$.

Theorem 3.1 *(Ballard [1], Jantzen [9]) For* $p \geq 2h-2$ *and* $\lambda \in X_n$ *we have*

$$T(\bar{\lambda})|_{\mathcal{U}_n} \cong Q(\lambda).$$

Donkin conjectures in [6](2.2) that Theorem 3.1 holds without any restriction on the prime p.

Theorem 3.2 *(Jantzen [10], Chastkofsky [3]) For* $\lambda \in X_n$*, the restriction of* $T(\bar{\lambda})$ *to* $kG(q)$ *is projective and* $U(\lambda)$ *occurs as a summand with multiplicity one.*

It turns out that the tilting modules appearing in Theorems 3.1 and 3.2 are projective and injective in the appropriate subcategory of the category of rational G-modules, called the category of p^n-bounded modules.

Proposition 3.1 *(Jantzen [9](section 4), [8](II.11.11)) Assume* $p \geq 2h-2$*. For* $\lambda \in X_n$*, the tilting module* $T(\bar{\lambda})$ *is the projective cover (and injective hull) of* $L(\lambda)$ *in the category of* p^n*-bounded* G*-modules i.e. the category of* G*-modules belonging to* π_1 *where*

$$\pi_1 = \{\mu \in X^+(T) \mid \langle \mu, \beta^v \rangle < 2p^n(h-1) \text{ for } \beta \in \Phi \cap X^+(T)\}.$$

This proposition is false when $p < 2h-2$ (see [9](4.6))

Remark: It should be noted that all the above results have been proved before the notion of tilting modules was introduced. So these results were given in terms of 'Humphreys-Verma component' of the G-module $St_n \otimes L((p^n-1)\rho + w_0\lambda)$, i.e. the indecomposable summand containing the highest weight $2(p^n-1)\rho + w_0\lambda$. But a result of Pillen [11] (see also [6](2.5)) tells us that this component is exactly the tilting module $T(\bar{\lambda})$.

Lemma 3.1 *For $\lambda \in X_n$ the quotient*

$$T(\bar{\lambda})/O^{\pi}(T(\bar{\lambda}))$$

is zero if $\lambda \notin \pi$ and for $\lambda \in \pi$ it is the projective cover $P_{\pi}(\lambda)$ of $L(\lambda)$ in the category of $S(\pi)$-modules.

Proof:
This follows from Proposition 3.1 and properties of the functor O^{π} (see section 1.3). QED

Let us first prove Theorem 2.1. For $\lambda \in X_n$, Lemma 3.1 gives an exact sequence of G-modules

$$0 \longrightarrow O^{\pi}(T(\bar{\lambda})) \longrightarrow T(\bar{\lambda}) \longrightarrow P_{\pi}(\lambda) \longrightarrow 0.$$

We use the convention that $P_{\pi}(\lambda) := 0$ when $\lambda \notin \pi$.
Restrict this exact sequence to $\mathcal{U}_n$. Then using Theorem 3.1, we get

$$0 \longrightarrow K(\lambda) \longrightarrow Q(\lambda) \longrightarrow P_{\pi}(\lambda) \longrightarrow 0$$

where $K(\lambda)$ denotes the restriction of $O^{\pi}(T(\bar{\lambda}))$ to $\mathcal{U}_n$. In order to define the map Φ_1, we need the following result.

Proposition 3.2 *Let $\lambda, \mu \in X_n$. If $\phi : Q(\lambda) \longrightarrow Q(\mu)$ is a $\mathcal{U}_e$-homomorphism then $\phi(K(\lambda)) \leq K(\mu)$.*

Proof:
Case 1: Suppose that ϕ is the restriction of a homomorphism of G-modules $\psi : T(\bar{\lambda}) \longrightarrow T(\bar{\mu})$ then by properties of the functor O^{π} we have that $O^{\pi}(T(\bar{\lambda})) \leq O^{\pi}(T(\bar{\lambda}))$ so we are done.

Case 2: Suppose now that ϕ is any $\mathcal{U}_n$-homomorphism. Consider the following diagram:

$$\begin{array}{ccccc} 0 \to (Q(\lambda), K(\mu))_{\mathcal{U}_n} & \to & (Q(\lambda), Q(\mu))_{\mathcal{U}_n} & \stackrel{\text{proj}}{\to} & (Q(\lambda), P_\pi(\mu))_{\mathcal{U}_n} \to 0 \\ & & \uparrow \text{res} & & \uparrow \text{res} \\ & & (T(\bar{\lambda}), T(\bar{\mu}))_G & \stackrel{\text{proj}}{\to} & (T(\bar{\lambda}), P_\pi(\mu))_G \to 0 \end{array}$$

(where we have omitted the Hom to gain space). Note that

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{U}_n}(Q(\lambda), P_\pi(\mu)) &= [P_\pi(\mu) : L(\lambda)] \\ &= \dim \operatorname{Hom}_{S(\pi_1)}(P_{\pi_1}(\lambda), P_\pi(\mu)) \\ &= \dim \operatorname{Hom}_G(T(\bar{\lambda}), P_\pi(\mu)) \end{aligned}$$

so the restriction map gives an isomorphism

$$\operatorname{Hom}_G(T(\bar{\lambda}), P_\pi(\mu)) \cong \operatorname{Hom}_{\mathcal{U}_n}(Q(\lambda), P_\pi(\mu)).$$

Consider the map $\text{proj} \circ \phi$, using the above isomorphism we can find a G-homomorphism $\psi : T(\bar{\lambda}) \longrightarrow T(\bar{\mu})$ such that $\text{proj} \circ \psi = \text{proj} \circ \phi$. This means that $\text{res}(\psi) = \phi$ modulo $\operatorname{Hom}_{\mathcal{U}_n}(Q(\lambda), K(\mu))$, i.e. there exists a homomorphism $\eta : Q(\lambda) \longrightarrow K(\mu)$ such that $\phi = \text{res}(\psi) + \eta$. In particular, using case 1, we get that $\phi(K(\lambda)) \leq K(\mu)$. QED

Now we can define Φ_1 by sending $\phi : Q(\lambda) \to Q(\mu)$ to

$$\bar{\phi} : Q(\lambda)/K(\lambda) = P_\pi(\lambda) \longrightarrow Q(\mu)/K(\mu) = P_\pi(\mu).$$

Since every $S(\pi)$-homomorphism $P_\pi(\lambda) \to P_\pi(\mu)$ can be viewed as a $\mathcal{U}_n$-homomorphism and the $Q(\lambda)$'s are projective $\mathcal{U}_n$-modules, we can lift it to a $\mathcal{U}_n$-homomorphism $Q(\lambda) \to Q(\mu)$. This proves that the algebra map Φ_1 is surjective and hence ends the proof of Theorem 2.1.

Let us now turn to finite group $G(q)$. The structure of the proof of Theorem 2.2 is exactly the same as for Theorem 2.1 but it is slightly more delicate as the restriction of the tilting module to the finite group $G(q)$ is not indecomposable in general. Note that for each $\lambda \in \pi$ the module $P_\pi(\lambda)$ is p^n-restricted i.e. all its composition factors $L(\mu)$ satisfy $\mu \in X_n$, so its structure does not change when restricted to $G(q)$ (see Lemma 4.2 below). As it has simple top isomorphic to $L(\lambda)$ we have an exact sequence of $G(q)$-modules

$$0 \longrightarrow N(\lambda) \longrightarrow U(\lambda) \longrightarrow P_\pi(\lambda) \longrightarrow 0$$

where $N(\lambda)$ denotes the kernel of the surjection $U(\lambda) \to P_\pi(\lambda)$.

Proposition 3.3 *Let $\lambda, \mu \in X_n$. If $\phi : U(\lambda) \longrightarrow U(\mu)$ is a $G(q)$-homomorphism then $\phi(N(\lambda)) \leq N(\mu)$.*

Proof:
As $T(\bar{\lambda})|_{G(q)}$ and $U(\lambda)$ are both projective, we have the following commutative diagrams.

$$\begin{array}{ccccc} & \exists j & U(\lambda) & & \\ & \swarrow & \downarrow & & \\ T(\bar{\lambda})|_{G(q)} & \longrightarrow & P_\pi(\lambda) & \longrightarrow 0 & \\ & & \downarrow & & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & \exists p & T(\bar{\mu})|_{G(q)} & & \\ & \swarrow & \downarrow & & \\ U(\mu) & \longrightarrow & P_\pi(\mu) & \longrightarrow 0 & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

If $\phi = p \circ \mathrm{res}(f) \circ j$ for some $f \in \mathrm{Hom}_G(T(\bar{\lambda}), T(\bar{\mu}))$ then using the above diagram and properties of the functor O^π we see that $\phi(N(\lambda)) \leq N(\mu)$. Now if ϕ is arbitrary, consider the following diagram

$$\begin{array}{cccccccc} 0 \to & (U(\lambda), N(\mu))_{G(q)} & \to & (U(\lambda), U(\mu))_{G(q)} & \to & (U(\lambda), P_\pi(\mu))_{G(q)} & \to 0 \\ & & & \eta \uparrow & & \epsilon \uparrow & \\ & & & (T(\bar{\lambda}), T(\bar{\mu}))_G & \to & (T(\bar{\lambda}), P_\pi(\mu))_G & \to 0 \end{array}$$

where $\eta : f \mapsto p \circ \mathrm{res}(f) \circ j$ and $\epsilon : f \mapsto \mathrm{res}(f) \circ j$.
Note that $\mathrm{Hom}_G(T(\bar{\lambda}), P_\pi(\mu)) = \mathrm{Hom}_{S(\pi_1)}(T(\bar{\lambda}), P_\pi(\mu))$ so we have

$$\dim \mathrm{Hom}_G(T(\bar{\lambda}), P_\pi(\mu)) = [P_\pi(\mu) : L(\lambda)] = \dim \mathrm{Hom}_{G(q)}(U(\lambda), P_\pi(\mu)).$$

We want to show that the map ϵ is one-to-one so that, by dimensions, it is an isomorphism. We need to prove that if $f \in \mathrm{Hom}_G(T(\bar{\lambda}), P_\pi(\mu))$ is non-zero then $\mathrm{res}(f) \circ j$ is non-zero. Consider the commutative diagram

$$\begin{array}{ccccc} U(\lambda) & \xrightarrow{j} & T(\bar{\lambda}) & \xrightarrow{f} & P_\pi(\mu) \\ & \searrow & \downarrow & & \\ & & P_\pi(\lambda) & & \end{array}$$

As $P_\pi(\mu)$ belongs to π, the quotient $T(\bar{\lambda})/\mathrm{Ker}\, f$ belongs to π as well and so $\mathrm{Ker}\, f \supseteq O^\pi(T(\bar{\lambda}))$. Thus we can define a map

$$\bar{f} : T(\bar{\lambda})/O^\pi(T(\bar{\lambda})) = P_\pi(\lambda) \longrightarrow P_\pi(\mu).$$

If f is non-zero then so is $\bar{f}$. Complete the above diagram to get the following commutative diagram

$$\begin{array}{ccccc} & j & & f & \\ U(\lambda) & \rightarrow & T(\bar{\lambda}) & \rightarrow & P_\pi(\mu) \\ & \searrow & \downarrow & \nearrow & \\ & & P_\pi(\lambda) & \bar{f} & \end{array}$$

If $\mathrm{res}(f) \circ j$ is zero then $\bar{f}$ must be zero but this is a contradiction. Now using the same argument as in the proof of Proposition 3.2 we see that $\phi(N(\lambda)) \leq N(\mu)$. QED.

This proposition allows us to define the map Φ_2 in the same way as we have defined Φ_1 and we see, using the fact that the $kG(q)$-modules $U(\lambda)$ are projective, that Φ_2 is a surjective algebra homomorphism. This ends the proof of Theorem 2.2.

4 Second proof

We now present a second proof of Theorems 2.1 and 2.2 due to Stephen Donkin which works without restriction on the prime p. We want to find surjective algebra homomorphisms

$$\mathcal{U}_n \longrightarrow S(\pi),$$
$$kG(q) \longrightarrow S(\pi).$$

But the algebra $S(\pi)$ was defined as the dual of the subcoalgebra $A(\pi)$ of the coordinate algebra $k[G]$ of the group G. So it is equivalent to find injective coalgebra homomorphisms

$$k[G_n] \longrightarrow A(\pi),$$
$$k[G(q)] \longrightarrow A(\pi).$$

There are natural candidates for these maps. The coalgebra $A(\pi)$ embeds in the coordinate algebra $k[G]$. In the first case we can compose this embedding with the projection $k[G] \to k[G_n] = k[G]/I^{[p^n]}$. In the second case, we can compose it with the map $k[G] \to k[G(q)]$ given by restriction of functions from G to $G(q)$. In the rest of this section we will show that the composition maps

$$\Psi_1 : A(\pi) \hookrightarrow k[G] \longrightarrow k[G]/I^{[p^n]} = k[G_n],$$
$$\Psi_2 : A(\pi) \hookrightarrow k[G] \longrightarrow k[G(q)]$$

are injective. Note that Ψ_1 is a homomorphism of $\mathcal{U}_n$-modules, so in order to show that it is injective, it is enough to show that it is injective on the $\mathcal{U}_n$-socle of $A(\pi)$. Similarly, Ψ_2 is a homomorphism of $kG(q)$-modules, so it is enough to show that it is injective on the $G(q)$-socle of $A(\pi)$.

The next two lemmas are well known (see [8]II.9.21 and [5]), but we include elementary proofs for completeness.

Lemma 4.1 *Let M be a p^n-restricted G-module then*

$$\mathrm{soc}_G(M) = \mathrm{soc}_{\mathcal{U}_n}(M).$$

Proof:
If $L(\mu) \in \mathrm{soc}_{\mathcal{U}_n}(M)$ then there exists a G-module U such that $L(\mu) \otimes U^F$ is a G-submodule of M. But M is restricted, so U must be trivial and $L(\mu)$ is a G-submodule of M. This proves that $\mathrm{soc}_{\mathcal{U}_n}(M) \subseteq \mathrm{soc}_G(M)$. The other inclusion is obvious. QED

Lemma 4.2 *Let M be a p^n-restricted G-module then*

$$\mathrm{soc}_G(M) = \mathrm{soc}_{G(q)}(M).$$

Proof:
Suppose first that M has composition length 2. Without loss of generality, we can assume that the root system of G is connected. The following argument is modeled on an argument of Anderson (reference to be added). Assume, for a contradiction, that M has simple G-socle and is semisimple as a $G(q)$-module. Replacing M by M^* if necessary, we can assume that $L \cong L(\lambda)$, $M/L \cong L(\mu)$ with $\lambda > \mu$. Thus M embeds in $\nabla(\lambda)$, but $\nabla(\lambda)$ embeds in

$St_n \otimes L((q-1)\rho + w_0(\lambda))$, so we have an embedding $M \hookrightarrow St_n \otimes L((q-1)\rho + w_0(\lambda))$. By assumption we have

$$\begin{aligned} &\mathrm{Hom}_{G(q)}(L(\mu), St_n \otimes L((q-1)\rho + w_0(\lambda))) \\ &= \mathrm{Hom}_{G(q)}(L(\mu) \otimes L((q-1)\rho - \lambda), St_n) \\ &\neq 0. \end{aligned}$$

Hence there exists a G-composition factor $L(\tau)$ of $L(\mu) \otimes L((q-1)\rho - \lambda)$ such that St_n is a $G(q)$-composition factor of $L(\tau)$. So we have $\tau \leq \mu + (q-1)\rho - \lambda$, and as $\lambda > \mu$ we have $\langle \tau, \alpha_0^v \rangle < (q-1)\langle \rho, \alpha_0^v \rangle$. But St_n must be a $G(q)$-composition factor of $L(\tau)$ and $L(\tau) = L(\nu_1) \otimes L(\nu_2)$ as a $G(q)$-modules. So we must have

$$(q-1)\langle \rho, \alpha_0^v \rangle \leq \langle \nu_1 + \nu_2, \alpha_0^v \rangle \leq \langle \tau, \alpha_0^v \rangle.$$

But this is a contradiction.

Now consider the general case and suppose that M is a counterexample of minimal length. Note that we can assume that M has simple G-socle. In fact if L and L' are two different simple G-submodules of M then we have an injection

$$\phi : M \hookrightarrow M/L \oplus M/L'.$$

Now by minimality,

$$\begin{aligned} &\mathrm{soc}_G(M/L) = \mathrm{soc}_{G(q)}(M/L) \\ \text{and} \quad &\mathrm{soc}_G(M/L') = \mathrm{soc}_{G(q)}(M/L'). \end{aligned}$$

Identifying M with $\phi(M)$ we see that

$$\begin{aligned} &\mathrm{soc}_G(M) = M \cap \mathrm{soc}_G(M/L \oplus M/L') \\ \text{and} \quad &\mathrm{soc}_{G(q)}(M) = M \cap \mathrm{soc}_{G(q)}(M/L \oplus M/L'). \end{aligned}$$

But the right hand sides coincide.

Assume $\mathrm{soc}_G(M) = L$. Suppose, for a contradiction, that $L \neq \mathrm{soc}_{G(q)}(M)$. Then M has a $G(q)$-submodule $Z = K \oplus L$ where $K \cong L(\mu)$ is simple. Take a $G(q)$-homomorphism $\eta : L(\mu) \to K$ giving rise to a homomorphism $\bar{\eta} : L(\mu) \to M/L$. Now, by minimality of M we have

$$\mathrm{Hom}_{G(q)}(L(\mu), M/L) = \mathrm{Hom}_G(L(\mu), M/L).$$

So the image of $\bar{\eta}$ is a G-submodule Z/L of M/L. Hence Z is a G-submodule of M, but it has length 2, so $\mathrm{soc}_G(Z) = \mathrm{soc}_{G(q)}(Z) = L \oplus K$. This contradicts the fact that M has simple socle. QED

As $A(\pi)$ is a p^n-restricted G-module, we have

$$\mathrm{soc}_{\mathcal{U}_n}(A(\pi)) = \mathrm{soc}_{G(q)}(A(\pi)) = \mathrm{soc}_G(A(\pi)).$$

Now $A(\pi)$ is a left G-module, so it can be viewed as a right $k[G]$-comodule. But it belongs to π, so it is a right $A(\pi)$-comodule.

Let us recall some standard facts about coalgebras and comodules. Let C be a coalgebra and assume that C is Schurian, i.e. $\mathrm{End}_C(L) = k$ for all simple C-comodule L. Let V be a right C-comodule with structure map $\tau : V \to V \otimes C$. The coefficient space of V, denoted by $\mathrm{cf}_C(V)$ is defined as follows. Let $\{v_i \,|\, i \in I\}$ be a basis for V then we have

$$\tau(v_i) = \sum_j v_j \otimes c_{ji}.$$

The coefficient space of V is the span of all the c_{ji} for $i, j \in I$. It does not depend on the choice of basis for V. The coalgebra C itself is a C-comodule with socle given by

$$\mathrm{soc}_C C = \bigoplus_{\lambda \in \Lambda} \mathrm{cf}_C(L(\lambda))$$

where $\{L(\lambda) \,|\, \lambda \in \Lambda\}$ is a complete set of non-isomorphic simple C-comodules. Moreover, we have

$$\mathrm{cf}_C(L(\lambda)) \cong L(\lambda)^{\dim L(\lambda)}.$$

If $\phi : C \to C'$ is a homomorphism of coalgebras, then we can turn V into a right C'-comodule via the structure map $\tau' = (1 \otimes \phi)\tau : V \to V \otimes C'$. Then by definition, we have that the coefficient space of V as a C'-module is given by

$$\mathrm{cf}_{C'}(V) = \phi(\mathrm{cf}_C(V)).$$

Applying the above remarks to our case we first see that

$$\mathrm{soc}_G(A(\pi)) = \mathrm{soc}_{A(\pi)}(A(\pi)) = \bigoplus_{\lambda \in \pi} \mathrm{cf}_{A(\pi)}(L(\lambda)).$$

Under the coalgebra homomorphisms Ψ_1 and Ψ_2 we get

$$\begin{aligned}
\Psi_1(\mathrm{soc}_{A(\pi)}(A(\pi))) &= \bigoplus_{\lambda \in \pi} \Psi_1(\mathrm{cf}_{A(\pi)}(L(\lambda))) \\
&= \bigoplus_{\lambda \in \pi} \mathrm{cf}_{k[G_n]}(L(\lambda)).
\end{aligned}$$

Similarly, we have that

$$\Psi_2(\mathrm{soc}_{A(\pi)}(A(\pi))) = \bigoplus_{\lambda\in\pi} \mathrm{cf}_{k[G(q)]}(L(\lambda)).$$

A dimension count now shows that Ψ_1, resp. Ψ_2, are injective on the $\mathcal{U}_n$-socle, resp. $kG(q)$-socle, of $A(\pi)$.

5 Remarks

When $G = SL_2(K)$ the set of p^n-restricted weights X_n is saturated in $X^+(T)$ and it is easy to see that the quasi-hereditary algebra $S(X_n)$ is isomorphic to the direct sum of Schur algebras $S_K(2, q-1) \oplus S_K(2, q-2)$.

In general, the set of p^n-restricted weights is not saturated. We want to find large saturated subsets of X_n. It is easy to see that the set C_n given by

$$C_n = \{\lambda \in X^+(T) \mid \langle \lambda, \beta^v \rangle < p^n \ \forall \beta \in \Phi \cap X^+(T)\}$$

is saturated in $X^+(T)$.

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