Extensions of modules for $SL(2, K)$

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In this paper, we consider the induced modules $\nabla$ and the Weyl modules $\Delta$ for the algebraic group $G = SL(2, K)$ where $K$ is an algebraically closed field of characteristic $p > 0$. We determine the $G$-modules $H^i(G_1, \nabla(s) \otimes \nabla(t))$ for all $i \geq 0$, where $G_1$ is the first Frobenius kernel of $G$. We then use it to find the Ext$^1$-spaces between twisted tensor products of Weyl modules and induced modules for $G$. Moreover, we describe explicitly the non-split extensions corresponding to $\nabla$'s.

Key words: special linear group, symmetric powers, decomposition matrix.
Introduction

In the theory of highest weight categories, the classes of modules $\nabla$ and $\Delta$ are of central interest. In particular, twisted tensor products of these modules occur as important subquotients of $\nabla$ and $\Delta$ (see [12] and [13]).

Here we consider these modules for the group $G = SL(2, K)$, the special linear group of dimension 2 over an algebraically closed field $K$ of characteristic $p > 0$.

Suppose that $F : G \rightarrow G$ is the corresponding Frobenius morphism and let $G_1$ denote the first Frobenius kernel of $G$. If $V$ is a $G$-module then we denote by $V^F$ its Frobenius twist. Considered as a $G_1$-module, $V^F$ is trivial. Conversely, if $W$ is a $G$-module on which $G_1$ acts trivially then $W \cong V^F$ for a unique $G$-module $V$ and we write $W^{(-1)} := V$.

Consider the Borel subgroup $B$ of $G$ consisting of lower triangular matrices and for $\lambda \in \mathbb{N}$, let $K_\lambda$ denote the 1-dimensional $B$-module of weight $\lambda$. Define the induced $G$-module $\nabla(\lambda)$ by

$$\nabla(\lambda) := \text{Ind}_{B}^{G}(K_\lambda).$$

This is isomorphic to the symmetric power $S^\lambda E$ where $E$ is the natural 2-dimensional $G$-module. The Weyl $G$-modules, $\Delta(\lambda)$, are defined by

$$\Delta(\lambda) := \nabla(\lambda)^\ast.$$

Note that $\text{soc}\nabla(\lambda) = \text{top}\Delta(\lambda) = L(\lambda)$ is simple and $\{L(\lambda), \lambda \in \mathbb{N}\}$ form a complete set of non-isomorphic simple $G$-modules. For $0 \leq \lambda \leq p - 1$ we have $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$ and in general Steinberg’s tensor product theorem tells us...
that if $\lambda = \sum_{i \geq 0} \lambda_ip^i$ is the $p$-adic expansion of $\lambda$ then $L(\lambda)$ is given by

$$L(\lambda) = \bigotimes_{i \geq 0} L(\lambda_i)^{p^i}.$$  

The simple $G$-modules are thus self-dual.

The modules $\nabla(\lambda)$ and $\Delta(\lambda)$ have highest weight $\lambda$ occurring with multiplicity 1 and all their other weights $\mu$ satisfy $\mu < \lambda$.

In order to prove our results, we use the Lyndon-Hochschild-Serre 5-term exact sequence relating the Ext$^1$-spaces of $G$ and $G_1$. For a rational $G$-module $V$, we have the exact sequence (see [3])

$$0 \rightarrow H^1(G,(V^G_1)^{(-1)}) \rightarrow H^1(G,V) \rightarrow H^1(G_1,V)^G \rightarrow H^2(G,(V^G_1)^{(-1)}) \rightarrow H^2(G,V).$$

In Section 1, we describe properties of $G_1$-modules and we compute Ext$^i_{G_1}(\Delta, \nabla)$ for $i \geq 0$ as $G$-modules. In Section 2, we use the 5-term exact sequence above and the results of Section 1 to compute Ext$^1_{G_1}(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t))$ for $0 \leq k, r$ and $0 \leq s, t \leq p^n - 1$. In particular, we show that it has at most dimension 1. We also find explicitly the non-split extensions corresponding to $\nabla$. This filtration of $\nabla$ by twisted tensor product of $\nabla$’s and $\Delta$’s explains the symmetries observed in the decomposition matrix of $G$.

1 Computing Ext$^i_{G_1}(\Delta, \nabla)$

The category of $G_1$-modules is equivalent to the category of $U$-modules where $U$ is the restricted Lie algebra of $G$. In particular, $U$ is a self-injective algebra
This category is very well understood ([9],[14]). The simple $U$-modules are the restriction of the $L(i)$ for $0 \leq i \leq p - 1$ and the corresponding projective $U$-modules $P(i)$ have the following structure: for $0 \leq i \leq p - 2$, $\text{soc}P(i) = \text{top}P(i) = L(i)$ and $\text{rad}P(i)/\text{soc}P(i) = L(j) \oplus L(j)$ where $i + j = p - 2$ and for $i = p - 1$ the projective module $P(p - 1) = L(p - 1)$ is simple. Thus the projective module $P(p - 1)$ is alone in its block and $P(i)$ and $P(j)$ belong to the same block if and only if $i = j$ or $i + j = p - 2$.

For an indecomposable non-projective $U$-module $M$, we denote by $\Omega(M)$ the kernel of the projective cover of $M$ (and we define inductively $\Omega^k(M) = \Omega(\Omega^{k-1}(M))$. Similarly, we define $\Omega^{-1}(M)$ to be the cokernel of the injective hull of $M$ (and we define inductively $\Omega^{-k}(M)$). The projective (injective) $G_1$-modules are restrictions of $G$-modules and for $n \geq 0$, we have an exact sequence of $G$-modules ([17],[4])

$$0 \longrightarrow \nabla(np + i) \longrightarrow P(i) \otimes \nabla(n)^F \longrightarrow \nabla((n + 1)p + j) \longrightarrow 0.$$ 

The restriction of this sequence to $G_1$ gives the projective cover of $\nabla((n + 1)p + j)$ and the injective hull of $\nabla(np + i)$.

The $G_1$-module $\nabla(np + i)$ has Loewy length 2 for $n \geq 1$. We have a sequence of $G$-modules ([17],[12])

$$0 \longrightarrow \nabla(n)^F \otimes \nabla(i) \longrightarrow \nabla(np + i) \longrightarrow \nabla(n - 1)^F \otimes \Delta(j) \longrightarrow 0 \quad (1)$$

and its restriction to $G_1$ gives the Loewy series of $\nabla(np + i)$ as a $G_1$-module.

Note finally that if $V$, $W$ and $X$ are $G$-modules and $n \geq 0$ then $\text{Ext}^n_{G_1}(V,W)$
has a natural structure of $G$-module and

$$\text{Ext}^n_{G_1}(V, W \otimes X^F) \cong \text{Ext}^n_{G_1}(V, W) \otimes X^F$$

as $G$-modules.

W. van der Kallen proved in [16] that if $V$ is a $G$-module with a good filtration (that is a filtration with quotients isomorphic to some $\nabla$’s) then $H^0(G_1, V)^{(-1)}$ has a good filtration and hence, by dimension shifting (see [7]),

$H^i(G_1, V)^{(-1)}$ has a good filtration for all $i \geq 0$. Note that the module $V = \nabla \otimes \nabla$ has a good filtration and the next two Propositions give the $G$-modules $H^i(G_1, V) = \text{Ext}^i_{G_1}(\Delta, \nabla)$ for $i \geq 0$.

Write $t = t_1p + t_0$ and $s = s_1p + s_0$ where $0 \leq s_0, t_0 \leq p - 1$.

**Proposition 1.1** For $i \geq 1$ we have

$$\text{Ext}^i_{G_1}(\Delta(s), \nabla(t)) \cong \begin{cases} 
\nabla(s_1 + t_1 + i)^F & \text{if } s_0 + t_0 = p - 2 \text{ and } i \text{ odd} \\
0 & \text{or } s_0 = t_0 \leq p - 2 \text{ and } i \text{ even}
\end{cases}$$

*Proof:*

From the block structure of $G_1$ we only need to consider the cases $s_0 = t_0$ and $s_0 + t_0 = p - 2$. Note that if $s_0 = t_0 = p - 1$ then $\Delta(s)$ and $\nabla(t)$ are projective and so there is no non-split extension. Now suppose $s_0, t_0 \leq p - 2$.

$$\text{Ext}^i_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \cong \text{Ext}^i_{G_1}(\Omega^{-s_1}(\Delta(s_1p + s_0), \Omega^{-s_1}(\nabla(t_1p + t_0)))$$
\[
\begin{cases}
\text{Ext}^1_{G_1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \\
\text{Ext}^1_{G_1}(\Delta(p - 2 - s_0), \nabla((s_1 + t_1)p + p - 2 - t_0))
\end{cases}
\]

if \( s_1 \) even

Now consider the exact sequence,

\[
0 \to \nabla((s_1 + t_1)p + t_0) \to P(t_0) \otimes \nabla(s_1 + t_1)^F \to \nabla((s_1 + t_1 + 1)p + p - 2 - t_0) \to 0
\]

and apply \( \text{Hom}_{G_1}(\Delta(s_0), -) \) to get

\[
0 \to \text{Hom}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \to \text{Hom}_{G_1}(\Delta(s_0), P(t_0) \otimes \nabla(s_1 + t_1)^F) \\
\text{---} \to \text{Hom}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1 + 1)p + p - 2 - t_0)) \\
\text{---} \to \text{Ext}^1_{G_1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \to 0
\]

and

\[
\text{Ext}^{i+1}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \cong \text{Ext}^i_{G_1}(\Delta(s_0), \nabla((s_1 + t_1 + 1)p + p - 2 - t_0)).
\]

Thus, if we prove the case \( i = 1 \) then the result follows by induction. Now, observe that in the exact sequence (2) the first two terms are isomorphic (\( \Delta(s_0) \) is simple and \( P(t_0) \otimes \nabla(s_1 + t_1)^F \) is the injective hull of \( \nabla((s_1 + t_1)p + t_0) \)), hence the last two terms are isomorphic too and we get

\[
\text{Ext}^1_{G_1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \\
\cong \text{Hom}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1 + 1)p + p - 2 - t_0)) \\
\cong \text{Hom}_{G_1}(\Delta(s_0), P(p - 2 - t_0) \otimes \nabla(s_1 + t_1 + 1)^F) \\
\cong \text{Hom}_{G_1}(\Delta(s_0), P(p - 2 - t_0)) \otimes \nabla(s_1 + t_1 + 1)^F \\
\cong \begin{cases}
\nabla(s_1 + t_1 + 1)^F & \text{if } s_0 + t_0 = p - 2 \\
0 & \text{otherwise}.
\end{cases}
\]
The proposition then follows by induction on $i$. QED

**Proposition 1.2**

$$\text{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 p + t_0)) \cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof:**

Note that by the decomposition into blocks of $G_1$, we only need to consider the cases $s_0 + t_0 = p - 2$ and $s_0 = t_0$. Suppose for a start that $s_0, t_0 \leq p - 2$.

Consider the exact sequence

$$0 \to \nabla(t_1)^F \otimes \nabla(t_0) \to \nabla(t_1 p + t_0) \to \nabla(t_1 - 1)^F \otimes \Delta(p - 2 - t_0) \to 0.$$

Apply $\text{Hom}_{G_1}(\Delta(s_1 p + s_0), -)$ to get the exact sequence

$$0 \to \text{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) \to \text{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 p + t_0))$$

$$\to \text{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 - 1)^F \otimes \Delta(p - 2 - t_0)) \to \text{Ext}^1_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1)^F \otimes \nabla(t_0))$$

$$\to \text{Ext}^1_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 p + t_0)). \quad (3)$$

Now,

$$\text{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) \cong \text{Hom}_{G_1}(\nabla(t_0), \nabla(s_1 p + s_0)) \otimes \nabla(t_1)^F$$

$$\cong \text{Hom}_{G_1}(\nabla(t_0), P(s_0)) \otimes \nabla(s_1)^F \otimes \nabla(t_1)^F$$

$$\cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F & \text{if } s_0 = t_0 \\ 0 & \text{otherwise,} \end{cases}$$

and
\[
\text{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1 - 1)^F \otimes \Delta(p - 2 - t_0))
\]
\[
\cong \text{Hom}_{G_1}(\nabla(p - 2 - t_0), \nabla(s_1p + s_0)) \otimes \nabla(t_1 - 1)^F.
\]
\[
\cong \text{Hom}_{G_1}(\nabla(p - 2 - t_0), P(s_0)) \otimes \nabla(s_1)^F \otimes \nabla(t_1 - 1)^F.
\]
\[
\cong \begin{cases} 
(\nabla(s_1) \otimes \nabla(t_1 - 1))^F & \text{if } s_0 + t_0 = p - 2 \\
0 & \text{otherwise}.
\end{cases}
\]

Using Proposition 1.1, we get

\[
\text{Ext}^1_{G_1}(\Delta(s_1p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) \cong \text{Ext}^1_{G_1}(\nabla(t_0), \nabla(s_1p + s_0)) \otimes \nabla(t_1)^F.
\]
\[
\cong \begin{cases} 
(\nabla(s_1 + 1) \otimes \nabla(t_1))^F & \text{if } s_0 + t_0 = p - 2 \\
0 & \text{otherwise.}
\end{cases}
\]

and

\[
\text{Ext}^1_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \cong \begin{cases} 
\nabla(s_1 + t_1 + 1)^F & \text{if } s_0 + t_0 = p - 2 \\
0 & \text{otherwise}.
\end{cases}
\]

So if \( s_0 + t_0 = p - 2 \) and \( p > 2 \) (i.e. \( s_0 \neq t_0 \)), then the exact sequence (3) becomes

\[
0 \longrightarrow \text{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \longrightarrow (\nabla(s_1) \otimes \nabla(t_1 - 1))^F
\]
\[
\longrightarrow (\nabla(s_1 + 1) \otimes \nabla(t_1))^F \longrightarrow \nabla(s_1 + t_1 + 1)^F.
\]

As

\[
\dim(\nabla(s_1 + 1) \otimes \nabla(t_1))^F = \dim(\nabla(s_1) \otimes \nabla(t_1 - 1))^F + \dim \nabla(s_1 + t_1 + 1)^F,
\]

we deduce that

\[
\text{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) = 0.
\]
If $s_0 = t_0$ and $p = 2$, the exact sequence (3) has the form

$$0 \longrightarrow (\nabla (s_1) \otimes \nabla (t_1))^F \longrightarrow \text{Hom}_{G_1} (\Delta (s_1^2 + s_0), \nabla (t_1^2 + t_0))$$

$$\longrightarrow (\nabla (s_1) \otimes \nabla (t_1 - 1))^F \longrightarrow (\nabla (s_1 + 1) \otimes \nabla (t_1))^F \longrightarrow \nabla (s_1 + t_1 + 1)^F.$$ 

Hence,

$$\text{Hom}_{G_1} (\Delta (s_1^2 + s_0), \nabla (t_1^2 + t_0)) \cong (\nabla (s_1) \otimes \nabla (t_1))^F.$$ 

Finally, if $s_0 = t_0$ and $p > 2$ then clearly

$$\text{Hom}_{G_1} (\Delta (s_1^p + s_0), \nabla (t_1^p + t_0)) \cong (\nabla (s_1) \otimes \nabla (t_1))^F.$$ 

In the case where $s_0 = t_0 = p - 1$, we have the following

$$\Delta (s_1^p + s_0) \cong \Delta (s_1)^F \otimes \Delta (p - 1)$$

$$\nabla (t_1^p + t_0) \cong \nabla (t_1)^F \otimes \nabla (p - 1),$$

and so

$$\text{Hom}_{G_1} (\Delta (s_1^p + (p - 1)), \nabla (t_1^p + (p - 1)))$$

$$\cong \text{Hom}_{G_1} (\Delta (p - 1), \nabla (p - 1)) \otimes (\nabla (s_1) \otimes \nabla (t_1))^F$$

$$\cong (\nabla (s_1) \otimes \nabla (t_1))^F.$$ 

This completes the proof. QED

2 Extensions of $G$-modules

In [5] and [8], Cox and Erdmann determined the Ext$^1$ and the Hom spaces between $\nabla (\lambda)$ and $\nabla (\mu)$ for arbitrary weights $\lambda$ and $\mu$. For completeness and
to fix our notation, we state their result here.

For $0 \leq a \leq p - 1$ denote by $\hat{a}$, the integer such that $a + \hat{a} = p - 1$. For a weight $\mu$, define

$$\psi^0(\mu) = \left\{ \sum_{i=0}^{u-1} \hat{\mu}_i p^i : u \geq 0 \right\}$$

and

$$\psi^1(\mu) = \left\{ \sum_{i=0}^{u-1} \hat{\mu}_i p^i + p^{u+a} : \hat{\mu}_u \neq 0, \ a \geq 1, \ u \geq 0 \right\} \cup \left\{ \sum_{i=0}^{u} \hat{\mu}_i p^i : \hat{\mu}_u \neq 0, \ u \geq 0 \right\}.$$ 

With this notation we have,

$$\text{Hom}_G(\n(\lambda), \n(\mu)) \cong \begin{cases} K & \text{if } \lambda = \mu + 2d, \ d \in \psi^0(\mu) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{Ext}^1_G(\n(\lambda), \n(\mu)) \cong \begin{cases} K & \text{if } \lambda = \mu + 2e, \ e \in \psi^1(\mu) \\ 0 & \text{otherwise} \end{cases}$$

In [2], Cline determined all the Ext$^1$-spaces between simple $G$-modules. In particular, for simple modules $\n(r)^F \otimes \n(s)$ and $\n(k)^F \otimes \n(t)$, he proved that

$$\text{Ext}^1_G(\n(r)^F \otimes \n(s), \n(k)^F \otimes \n(t)) \cong \begin{cases} K & \text{if } r = k \pm 1, \ s + t = p - 2 \\ 0 & \text{otherwise} \end{cases}$$

The following theorem extends this result.
Theorem 2.1 \textit{Let} $0 \leq k, r$ \textit{and} $0 \leq s, t \leq p^n - 1$ \textit{then we have}

$$\text{Ext}_G^1(\nabla(r)^F \otimes \Delta(s), \nabla(k)^F \otimes \nabla(t)) \cong \begin{cases} K & \text{if } r = k + 2e, \ e \in \psi^1(k) \\ s = t & \text{if } r = k \pm 1 + 2d, \ d \in \psi^0(k) \\ r = k \pm 1 + 2d, \ d \in \psi^0(k) & \text{or } t = t_0 + t_1p^i, \ 0 \leq t_0 \leq p^i - 1 \\ s = t_0 + (p^{n-1} - 2 - t_1)p^i & \\ 0 & \text{otherwise} \end{cases}$$

\textit{Proof:}

In order to prove this theorem, we use the five terms exact sequence:

$$0 \longrightarrow H^1(G, (V^{G_1})^{-1}) \longrightarrow H^1(G, V) \longrightarrow H^1(G, V)^G \longrightarrow H^2(G, (V^{G_1})^{-1}) \longrightarrow H^2(G, V),$$

with $V = \Delta(r)^F \otimes \nabla(k)^F \otimes \nabla(s) \otimes \nabla(t)$.

Write $s = s_1p + s_0$ and $t = t_1p + t_0$. Let us first compute $H^1(G, (V^{G_1})^{-1})$.

Using Proposition 1.2, we have

$$V^{G_1} = \text{Hom}_{G_1}(\Delta(s), \nabla(t)) \otimes \Delta(r)^F \otimes \nabla(k)^F \cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F \otimes \Delta(r)^F \otimes \nabla(k)^F & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$(V^{G_1})^{-1} \cong \begin{cases} \nabla(s_1) \otimes \nabla(t_1) \otimes \Delta(r)^{F^n-1} \otimes \nabla(k)^{F^n-1} & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases}$$
Hence for $s_0 = t_0$ we have

$$H^1(G, (V^G_1)^{(-1)}) \cong \text{Ext}^1_G(\nabla(r)^{F-1} \otimes \Delta(s_1), \nabla(k)^{F-1} \otimes \nabla(t_1)),$$

and is zero in all other cases.

Let us now compute $H^1(G_1, V^G)$. Using Proposition 1.1, we have

$$H^1(G_1, V^G) = \text{Ext}^1_{G_1}(\nabla(r)^F \otimes \Delta(s), \nabla(k)^F \otimes \nabla(t))$$

$$\cong \text{Ext}^1_{G_1}(\Delta(s), \nabla(t)) \otimes \Delta(r)^F \otimes \nabla(k)^F$$

$$\cong \begin{cases} 
\nabla(s_1 + t_1 + 1)^F \otimes \Delta(r)^F \otimes \nabla(k)^F & \text{if } s_0 + t_0 = p - 2 \\
0 & \text{otherwise.}
\end{cases}$$

Thus,

$$H^1(G_1, V^G) \cong \begin{cases} 
\text{Hom}_G(\Delta(s_1 + t_1 + 1)^F, \Delta(r)^F \otimes \nabla(k)^F) & \text{if } s_0 + t_0 = p - 2 \\
0 & \text{otherwise.}
\end{cases}$$

Note that all the weights of $\Delta(r)^F \otimes \nabla(k)^F$ are multiples of $p^n$, so to get non-zero homomorphisms, we must have $s_1 + t_1 + 1 = cp^{n-1}$ for some $c$. But $s, t \leq p^n - 1$ implies that $s_1 + t_1 \leq 2p^{n-1} - 2$, thus $c = 1$ and $s_1 + t_1 + 1 = p^{n-1}$.

Observe that

$$\text{Hom}_G(\Delta(p^{n-1})^F, \Delta(r)^F \otimes \nabla(k)^F) \cong \text{Hom}(\nabla(r)^F, \nabla(p^{n-1})^F \otimes \nabla(k)^F)$$

and that all the weights of $\nabla(r)^F$ are multiple of $p^n$ so the image of a homomorphism from $\nabla(r)^F$ to $\nabla(p^{n-1})^F \otimes \nabla(k)^F$ lies in the submodule $\nabla(1)^F \otimes \nabla(k)^F \leq \nabla(p^{n-1})^F \otimes \nabla(k)^F$. Hence,

$$\text{Hom}_G(\Delta(p^{n-1})^F, \Delta(r)^F \otimes \nabla(k)^F) \cong \text{Hom}_G(\nabla(r)^F, \nabla(1)^F \otimes \nabla(k)^F)$$

$$\cong \text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)).$$
We claim that $\text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) \cong K$ if $r = k \pm 1 + 2d$ where $d \in \psi^0(k)$ and zero otherwise. Consider the exact sequence

$$0 \longrightarrow \nabla(z - 1) \longrightarrow \nabla(1) \otimes \nabla(z) \longrightarrow \nabla(z + 1) \longrightarrow 0.$$  \hfill (6)

This sequence splits if and only if $z \neq -1 \pmod{p}$. Note that for $\text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k))$ to be non-zero, we must have $r + k = 1 \pmod{2}$. Now suppose $k = -1 \pmod{p}$ then we can assume $r \neq -1 \pmod{p}$ and so using (6) with $z = r$ we have

$$\text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) \cong \text{Hom}_G(\nabla(1) \otimes \nabla(r), \nabla(k)) \cong \text{Hom}_G(\nabla(r - 1) \otimes \nabla(r + 1), \nabla(k)),$$

Now, using (4) we deduce that $\text{Hom}_G(\nabla(r - 1), \nabla(k)) \cong K$ if and only if $r - 1 = k + 2d$ where $d \in \psi^0(k)$ and it is zero otherwise, and $\text{Hom}_G(\nabla(r + 1), \nabla(k)) \cong K$ if and only if $r + 1 = k + 2d'$ where $d' \in \psi^0(k)$ and zero otherwise. Suppose they are both non-zero then $k + 1 + 2d = k - 1 + 2d'$. But this can only happen when $d = 0$, $d' = 1$ and $r = k + 1$. This means that $k = p - 2 \pmod{p}$ and $r = -1 \pmod{p}$ contradicting our assumption. Now if $k \neq -1 \pmod{p}$ we use (6) with $z = k$ and the claim follows by a similar argument.

Hence, we have proved the following

$$H^1(G_1, V)^G \cong \begin{cases} K & \text{if } s_0 + t_0 = p - 2, s_1 + t_1 = p^{n-1} - 1 \\ r \pm 1 + 2d & \text{where } d \in \psi^0(k) \\ 0 & \text{otherwise.} \end{cases}$$
Let us now use the five term sequence to determine $H^1(G,V)$. We shall do this by induction on $n$. For $n = 1$ we have $s, t \leq p - 1$ and

$$H^1(G_1, (V^G_1)^{(1)}) \cong \begin{cases} 
K & \text{if } r = k + 2e, e \in \psi^1(k) \text{ and } s = t \\
0 & \text{otherwise}
\end{cases}$$

and

$$H^1(G_1, V^G) \cong \begin{cases} 
K & \text{if } r = k \pm 1 + 2d, d \in \psi^0(k) \text{ and } s + t = p - 2 \\
0 & \text{otherwise,}
\end{cases}$$

thus,

$$H^1(G, V) \cong \begin{cases} 
K & \text{if } r = k + 2e, e \in \psi^1(k) \text{ and } s = t \\
on r = k \pm 1 + 2d, d \in \psi^0(k) \text{ and } s + t = p - 2 \\
0 & \text{otherwise.}
\end{cases}$$

Now we use induction. Note that if $p = 2$ and $s_0 = t_0 = 0$ and $s_1 + t_1 = 2^{n-1} - 1$ then $\Delta(s_1)$ and $\nabla(t_1)$ are in different blocks of $G_1$ and so

$$\text{Ext}_G^i(\nabla(r)^{p^n-1} \otimes \Delta(s_1), \nabla(k)^{p^n-1} \otimes \nabla(t_1)) = 0 \quad \text{for all } i.$$
So for all prime $p$ we get

$$H^1(G, V) \cong \begin{cases} K & \text{if } r = k + 2e, \; e \in \psi^1(k) \\ s = t & \\ r = k \pm 1 + 2d, \; d \in \psi^0(k) & \text{or } t = t_0 + t_1 p^i, \; 0 \leq t_0 \leq p^i - 1 \\ s = t_0 + (p^{n-i} - 2 - t_1)p^i & \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of our theorem. QED

Note that if we set $n = 0$ and $s = t = 0$ in Theorem 2.1 we get Erdmann and Cox’s result given by equation (5).

The following proposition shows that when $r = k - 1$ and $s = p^n - 2 - t$, the extension is given by $\nabla(kp^n + t)$. By considering weights, it is easy to see that no other extension described in Theorem 2.1 can be isomorphic to an induced module $\nabla(\lambda)$.

**Proposition 2.1** For $k \in N$ and $0 \leq t \leq p^n - 2$, there is an exact sequence of $G$-modules

$$0 \longrightarrow \nabla(k)^{F^n} \otimes \nabla(t) \longrightarrow \nabla(kp^n + t) \longrightarrow \nabla(k - 1)^{F^n} \otimes \Delta(p^n - 2 - t) \longrightarrow 0.$$

Moreover, $\nabla(kp^n + t)$ is the only non-split extension, up to isomorphism, of $\nabla(k - 1)^{F^n} \otimes \Delta(p^n - t - 2)$ by $\nabla(k)^{F^n} \otimes \nabla(t)$. 

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Dually, the only non-split extension, up to isomorphism, of $\Delta(k)^F \otimes \Delta(t)$ by $\Delta(k-1)^F \otimes \nabla(p^n - t - 2)$ is given by $\Delta(kp^n + t)$.

**Remark 1:** For $k \in N$ we have an isomorphism between $\nabla(k-1)^F \otimes St_n$ and $\nabla(kp^n - 1)$ given by multiplication of polynomials. It is known that there is an isomorphism between these modules more generally, see for example [11](II.3).

**Proof of Proposition 2.1:**
If $n = 1$ then we are done by (1) (Section 1). Suppose $n > 1$ and write $t = ap^{n-1} + d$, for $0 \leq a \leq p - 1$ and $0 \leq d \leq p^{n-1} - 1$. Using induction we have an exact sequence

$$0 \rightarrow \nabla(kp + a)^{F_{n-1}} \otimes \nabla(d) \rightarrow \nabla((kp + a)p^{n-1} + d) \rightarrow \nabla(kp + (a - 1))^{F_{n-1}} \otimes \Delta(p^{n-1} - d - 2) \rightarrow 0.$$ 

Using the exact sequences (1) for $\nabla(kp + a)^{F_{n-1}}$ and $\nabla(kp + (a - 1))^{F_{n-1}}$ we get a filtration of $\nabla(kp^n + ap^{n-1} + d)$ with quotients

$$\nabla(k-1)^F \otimes \Delta(p - a - 1)^{F_{n-1}} \otimes \Delta(p^{n-1} - d - 2)$$
$$\nabla(k)^F \otimes \nabla(a - 1)^{F_{n-1}} \otimes \Delta(p^{n-1} - d - 2)$$
$$\nabla(k-1)^F \otimes \Delta(p - a - 2)^{F_{n-1}} \otimes \nabla(d)$$
$$\nabla(k)^F \otimes \nabla(a)^{F_{n-1}} \otimes \nabla(d)$$

Observe that the module $\nabla(kp^n + ap^{n-1} + d)$ is multiplicity-free, so that the four quotients have disjoint sets of weights. Hence, $\nabla(kp^n + t)/\nabla(k)^F \otimes \nabla(t)$
has a filtration with quotients

$$\nabla(k-1)^{F_n} \otimes \Delta(p-a-1)^{F_n-1} \otimes \Delta(p^{n-1} - d - 2)$$

$$\nabla(k-1)^{F_n} \otimes \Delta(p-a-2)^{F_n-1} \otimes \nabla(d)$$

Note that for $a = p - 1$ or $d = p^{n-1} - 1$, we only have one factor appearing and so we are done by Remark 1 above. So suppose $a \leq p - 2$ and $d \leq p^{n-1} - 2$.

Using a very similar argument to the proof of Theorem 2.1 we can show that

$$\text{Ext}_G^1(\nabla(k-1)^{F_n} \otimes \Delta(p-a-1)^{F_n-1} \otimes \Delta(p^{n-1} - d - 2),$$

$$\nabla(k-1)^{F_n} \otimes \Delta(p-a-2)^{F_n-1} \otimes \nabla(d)) \cong K.$$ 

Now as $\nabla(kp^n + t)$ has simple top (see [1]), $\nabla(kp^n + t)/\nabla(k)^{F_n} \otimes \nabla(t)$ cannot be a direct sum of non-zero modules. By induction, we know that $\Delta(p^n - ap^{n-1} - d - 2)$ has a filtration with quotients

$$\Delta(p-a-1)^{F_n-1} \otimes \Delta(p^{n-1} - d - 2)$$

$$\Delta(p-a-2)^{F_n-1} \otimes \nabla(d)$$

We deduce that the quotient $\nabla(kp^n + t)/\nabla(k)^{F_n} \otimes \nabla(t)$ is isomorphic to

$$\nabla(k-1)^{F_n} \otimes \Delta(p^n - ap^{n-1} - d - 2) = \nabla(k-1)^{F_n} \otimes \Delta(p^n - 2 - t).$$

This completes the proof QED

**Remark 2:** S. Donkin suggested an alternative proof of Proposition 2.1. I shall sketch his argument here. Let us start with the exact sequence of $B$-modules

$$0 \longrightarrow \nabla(s - 1) \otimes K_{-1} \longrightarrow \nabla(s) \longrightarrow K_1$$

(7)
for any positive integer $s$. Apply the Frobenius morphism $F^n$ to the sequence (7) and tensor it with $K_r$ for some $0 \leq r \leq p^n - 1$. Then applying the induction functor from $B$-modules to $G$-modules and using the duality of induction (see [11],II.4) gives the required sequence.

**Remark 3:** The composition factors of the $\nabla$'s are known for $SL(2, K)$ (use for example equation (1) repeatedly) but Proposition 2.1 gives a direct explanation of the symmetries observed by A. Henke in the decomposition matrix of $SL(2, K)$ (see [10]). More precisely, if we write $\lambda = kp^n + t$ with $k \leq p - 1$ then our proposition tells us that

$$[
abla(kp^n + t) : L(kp^n + a)] = [
abla(t) : L(a)],$$

$$[
abla(kp^n + t) : L((k - 1)p^n + b)] = [
abla(p^n - 2 - t) : L(b)].$$

Let us write the decomposition matrix of $G$ with the $\nabla$'s on the horizontal axis and the $L$'s on the vertical axis (see figures 1 and 2 below). Then for each $n \geq 1$ and each $1 \leq k \leq p - 1$, the columns corresponding to $\nabla(kp^n + t)$ for $0 \leq t \leq p^n - 1$ are obtained from the left bottom $p^n \times p^n$ block by

1. Translation of length $k$ along the diagonal,

2. Translation of length $k - 1$ along the diagonal and then reflection through the column corresponding to $\nabla(kp^n - 1)$ (shaded on the figures).

Hence, we can construct the decomposition matrix inductively starting with the left bottom $p \times p$ block which is just a diagonal matrix, as for $0 \leq r \leq p - 1$ we have $\nabla(r) = L(r)$. 19
Acknowledgements

I would like to thank my DPhil supervisor Karin Erdmann for her help and encouragement. I also thank Stephen Donkin for his helpful comments. This research was supported by the Fondation Wiener-Anspach (Brussels).

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